

Chapter 2. Basic Structures: sets, functions, sequences, and sums

2.1 Sets

definition: a *set* is an unordered collection of objects. e.g., $A = \{a_1, a_2, \dots, a_{10}\}$

elements in the set, $a \in A$,

elements not in the set: $b \notin B$

More examples of sets

Two sets are *equal* if they have the same elements.

$$\{a, b\} = \{b, a, b\}$$

Venn diagram, empty set (null set), ϕ

subset, $A \subseteq B$ if and only if
 $\forall x(x \in A \rightarrow x \in B)$

$\emptyset \subseteq S, S \subseteq S$

$A = B$ if and only
 $A \subseteq B$ and $B \subseteq A$

$A \subset B$
if $A \subseteq B$ and $\exists b(b \in B \wedge b \notin A)$

$|S|$, the cardinality the number of different elements in a set S

set builder $A = \{a : Q(a)\}$

where Q is a predicate

infinite sets, e.g., $N = \{x : x \geq 0 \text{ and } x \text{ is integer}\}$

power set of S is the set of all subsets of S

$$P(S) = \{A : A \subseteq S\}$$

examples:

$$P(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$$

$$P(\phi) = \{\phi\}$$

$$P(\{\phi\}) = \{\phi, \{\phi\}\}$$

$|P(S)| = ?$ for a finite set S

$$P(N) = ?$$

Cartesian product

ordered n-tuple: (a_1, a_2, \dots, a_n) ,

(a, b) is an ordered pair

$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$

if and only $a_i = b_i$ for $i = 1, 2, \dots, n$

$A \times B = \{(a, b) : a \in A \wedge b \in B\}$

examples, records, database tables

set notations with quantifiers

$\forall x \in S P(x)$

what is $A \times \phi$?

2.2 Set operations

various ways to combine two or more sets

union: $A \cup B = \{x : x \in A \vee x \in B\}$

intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$

A and B are *disjoint* if $A \cap B = \phi$

difference: $A - B = \{x : x \in A \wedge x \notin B\}$

complement: $\bar{A} = U - A$,
where U is the *universal set*.

Venn diagram representations

Set identities

table 1, p124. all can be proved by the definitions of sets and set operations

generalized unions and intersections

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^n A_i = \{x : \exists i x \in A_i\}$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^n A_i = \{x : \forall i x \in A_i\}$$

computer representations of sets

binary bits

union and intersection operations

2.3 Functions

a *function* from set A to set B is an association of exactly one element of B to each element in A , $f(a) = b$

This can be defined using proposition

$$\exists b \in B [f(a) = b \wedge \forall c (c \neq b \rightarrow f(a) \neq c)]$$

denoted: $f : A \longrightarrow B$

domain, co-domain, image, pre-image, map

addition of two functions: $f + g$

$$(f + g)(x) = f(x) + g(x)$$

product of two functions: fg

$$(fg)(x) = f(x)g(x)$$

function on subsets: if $S \subseteq A$

$f(S) = \{f(x) : x \in S\}$, images of S

$f(S) \subseteq B$

1-1 function (or *injective*): if

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y)$$

Is $f(x) = x^2$ a 1-1 function?

onto function (or *surjective*): if

$$\forall y \exists x (f(x) = y)$$

1-1 correspondence (or *bijective*) if

(1) 1-1

(2) onto

increasing, strictly increasing, decreasing, strictly decreasing functions.

inverse function of a function f
 $f^{-1}(b) = a$ when $f(a) = b$

conditions?

so $f^{-1}(f(a)) = a$

composite function of two functions f and g
 $(f \circ g)(a) = f(g(a))$

conditions?

Other representations: tables, graphs,

Floor function $\lfloor x \rfloor$

Ceiling function $\lceil x \rceil$

table 1, p144, useful properties of the Floor and Ceiling functions

2.4 Sequences and summations

A *sequence* is an ordered list of objects.

example:

1, 1, 2, 3, 5, 8, 13, ...

A sequence is a function: $N \rightarrow S$

The n th element in the sequence is a_n and the sequence is denoted as $\{a_n\}$.

example: consider sequence $\{a_n\}$ where

$a_n = 1/n$, the sequence is actually

1, 1/2, 1/3, ... called *harmonic sequence*.

arithmetic progression:

$$a_0 = a$$

$$a_1 = a + d$$

$$a_2 = a + 2d$$

...

$$a_n = a + nd$$

...

where $a_n - a_{n-1} = d$

geometric progression:

$$a_0 = a$$

$$a_1 = ar$$

$$a_2 = ar^2$$

...

$$a_n = ar^n$$

...

where $a_n/a_{n-1} = r$

summations:

$$\sum_{i=1}^n i = 1 + 2 + \dots + n = n(n+1)/2, \text{ but why?}$$

Generally:

$$\sum_{i=0}^n (a + id)$$

$$= (n+1)a + \sum_{i=0}^n id$$

$$= (n+1)a + d \sum_{i=0}^n i$$

$$= (n+1)a + dn(n+1)/2$$

$$\sum_{i=0}^n r^i = 1 + r + r^2 + \dots + r^n$$
$$= (r^{n+1} - 1)/(r - 1), \text{ but why?}$$

Generally:

$$\sum_{i=0}^n ar^i$$

$$= a \sum_{i=0}^n r^i$$

$$= a(r^{n+1} - 1)/(r - 1).$$

summations with lower and upper limits

$$\sum_{i=m}^n a_i = \sum_{i=0}^n a_i - \sum_{i=0}^{m-1} a_i$$

e.g.,

$$\sum_{i=11}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{10} i$$

$$= 20(20+1)/2 - 10(10+1)/2 = \frac{10}{2}(2 \times 21 - 11)$$

summation of infinite sequences

for $0 < r < 1$,

$$\sum_{i=0}^{\infty} r^i = 1 + r + r^2 + \dots + r^n \dots$$

$$= \lim_{n \rightarrow \infty} (r^{n+1} - 1)/(r - 1) = 1/(1 - r) \text{ when } r < 1$$

So for $r = 1/2$ the sum = 2.

More useful summation formulae: table 2, page 157.

Cardinality of sets

A set is *countable* if there is a one-to-one correspondence between this set and the set of positive integers (or the natural number set).

e.g., the set I of all integers is countable.

Proof: We build a function $f : I \rightarrow N$ from the set of integers to the set of natural numbers such that

$$f(x) = 0 \text{ if } x = 0$$

$$f(x) = 2x - 1 \text{ if } x > 0$$

$$f(x) = -2x \text{ if } x < 0$$

We need to show f is a one-to-one correspondence.

(1) proof of f being 1-1.

(2) proof of f being onto.

(1) proof of f being 1-1:

This is to show that for any $x \in I$ and $y \in I$,
 $f(x) = f(y) \rightarrow x = y$.

We have 9 combinations of $f(x)$ and $f(y)$:

a) $f(x) = 0, f(y) = 0: f(x) = f(y) \rightarrow x = y = 0$

b) $f(x) = 0, f(y) = 2y - 1: f(x) = f(y) \rightarrow y = \frac{1}{2}$ impos-
sible case

c) $f(x) = 0, f(y) = -2y: f(x) = f(y) \rightarrow y = 0 = x$

d) $f(x) = 2x - 1, f(y) = 0: \text{same as b)}$

e) $f(x) = 2x - 1, f(y) = 2y - 1: f(x) = f(y) \rightarrow x = y$

f) $f(x) = 2x - 1, f(y) = -2y: f(x) = f(y) \rightarrow \text{odd} = \text{even}$
impossible case.

g) $f(x) = -2x, f(y) = 0; \text{same as c)}$

h) $f(x) = -2x, f(y) = 2y - 1: \text{same as f)}$

i) $f(x) = -2x, f(y) = -2y: f(x) = f(y) \rightarrow x = y$

(2) proof of f being onto:

This is to show, for any $y \in N$, there is an $x \in I$ such that $f(x) = y$.

We have 3 scenarios for y :

a) $y = 0$: then we pick $x = 0$ such that $f(x) = y$.

b) y is odd: let $y = 2k + 1$ odd for some k ;
let $2x - 1 = 2k + 1$, then we pick $x = (2k + 2)/2 = k + 1$
such that $f(x) = y$

c) y is even: let $y = 2k$ even for some k ;
let $-2x = 2k$, then we pick $x = -k$ such that $f(x) = k$

Done!

e.g., the set of all positive rational numbers is countable.

e.g., the set R of real numbers is not countable.