Chapter 7. Advanced Counting Techniques

7.1 Recurrence Relations

We study functions or sequences that can be recursively defined.

Basic counting principles:

A *recurrence relation* for sequence $\{a_n\}$ is an equation that expresses a_n in terms of one of more of the previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, for some non-negative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relations.

Examples:

$$a_1 = 1, \quad a_n = 2a_{n-1},\ n > 1.$$

Solution to this recurrence: $1, 2, 2^2, 2^3, \ldots,$

Modeling with recurrence relations

modeling problems (or solutions):

example: Fibonacci sequence (just sequence)

example: running time of recursive algorithms

binary search

insertion sort

quick sort

example: compound interest (solution)

example: Tower of Hanoi (solution)

Solving recurrence relations using iteration techniques

example: solving $a_1 = 1 \;\; a_n = 2a_{n-1}$

example: solving time function for binary search

$$B(n) = B(n/2) + 1$$

example: solving time function for the ideal case of quick sort

$$Q(n) = 2Q(n/2) + n$$

example: solving time function for the average case of quick sort

$$Q(n) = Q(n/3) + Q(2n/3) + n$$

7.2 Solving Linear Recurrence Relations

First we look at recurrence relations that are *linear*, *homogeneous*.

Linear: left term is the sum of terms in the right (terms prior to the left term in the sequence)

$$H_n = 2H_{n-1} + 1$$

homogeneous: all terms in the right are terms preceding the left term

$$F_n = F_{n-1} + F_{n-2}$$

We explain the technique using Example:

$$a_0 = 2, a_1 = 7,\ a_n = a_{n-1} + 2a_{n-2}$$

The basic approach is to look for solutions of the form: $a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution for recurrence if and only if

$$r^n = r^{n-1} + 2r^{n-2}$$

we have

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

divided by r^{n-2},

$$r^2 - r - 2 = 0 \quad \text{characteristic equation}$$

solve it:

$$(r-2)(r+1) = 0$$

two roots: $r_1 = 2$ and $r_2 = -1$

Try $a_n = \beta_1 r_1^n + \beta_2 r_2^n$.

$a_0 = 2 = \beta_1 \times 1 + \beta_2 \times 1$
$a_1 = 7 = \beta_1 \times 2 + \beta_2 \times (-1)$

Solve $\beta_1 = 3$, $\beta_2 = -1$

So: $a_n = 3 \times 2^n - (-1)^n$.

Theorem 1 Let c_1, c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$, where $a_n = \beta_1 r_1^n + \beta_2 r_2^n$ is a solution for recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

where β_1, β_2 are determined by the initial conditions of the recurrence.

Consider the previous example:

$$a_0 = 2, a_1 = 7,\ a_n = a_{n-1} + 2a_{n-2}$$

here $c_1 = 1, c_2 = 2$, plug into the theorem to solve it.

Note that changing the values of a_0, a_1 may completely change the sequence, given the same recurrence.

E.g., $a_0 = 1$, $a_1 = 2$. Then

$a_0 = 1 = \beta_1 \times 1 + \beta_2 \times 1$
$a_1 = 1 = \beta_1 \times 2 + \beta_2 \times (-1)$

solve them: $\beta_1 = 1$, $\beta_2 = 1/3$ so

$a_n = 2^n + 1/3(-1)^n$, a different sequence.

Solving a recurrence

step 1: find out c_1, c_2;

step 2: solving the quadratic equation to get roots r_1 and r_2;

step 3: define $a_n = \beta_1 r_1^n + \beta_2 r_2^n$

step 4: solving β_1, β_2 using initial conditions.

Theorem 2 Let c_1, c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has a unique roots r. Then the sequence $\{a_n\}$, where $a_n = \beta_1 r^n + \beta_2 n r^n$ is a solution for recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

where β_1, β_2 are determined by the initial conditions of the recurrence.

Example: $a_n = 2a_{n-1} - a_{n-2}$

try different initial conditions:

(1) $a_0 = a_1 = 1$;

(2) $a_0 = 1, a_1 = 2$;

(3) $a_0 = 2, a_1 = 5$;