The Asymptotic Hadamard Conjecture

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Co-authors: Jason Gao, Catherine Greenhill, Brendan McKay, Bob Robinson
Host institution: Australia National University

Authors: Warwick de Launey & David A. Levin
(hopefully they are future co-authors)

NSA Mathematical Sciences Program

Main Definitions

An $n \times t$ matrix over $\{\pm 1\}$ is a partial Hadamard matrix provided the rows are pairwise orthogonal.

Definition: $H_{nt} :=$ the # of $n \times t$ partial Hadamard matrices

Example. One of the matrices counted by H_{58}:

$$\begin{matrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \\ - & - & + & + & + & + & - & - \end{matrix}$$

Example illustrates: $n \geq 3 \;\&\; H_{nt} \neq 0 \implies 4|t$

The Problem

Find an asymptotic formula for H_{nt} valid certain infinite sequences of pairs (n, t).

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Theorem (de Launey & Levin, 2010) Let (n, t) be an infinite sequence of pairs satisfying $4|t$ and $t \geq n^{12+\epsilon}$. Then along this sequence

$$H_{nt} \sim \frac{2^{nt+(n-1)^2}}{(2\pi t)^{d/2}}, \quad d = \binom{n}{2}.$$

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A Fourier-analytic approach to counting partial Hadamard matrices

Cryptography and Communications 2(2010) pp 307–334.

The Hadamard Conjecture

The HADAMARD CONJECTURE states that there exist square Hadamard matrices of size $n \times n$ for $n \in \{1, 2, 4, 8, 12, 16, 20, \ldots\}$.

Various constructions have been found

$n = 668$ is the first undecided value

$n = 428$ was decided in 2004

Outline

Circle Method Estimates
Short digression: Latin rectangles
The de Launey - Levin Theorem
An extension

The Circle Method

$$
\begin{aligned}
a_n &= [z^n] f(z) \\
&= \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz \\
&= \frac{1}{2\pi r^n} \int_{-\pi}^{+\pi} \frac{f(re^{i\theta})}{e^{ni\theta}} d\theta \\
&= \frac{1}{2\pi r^n} \left[\int_{-\delta}^{+\delta} \cdots \; + \; \int_{\delta \le |\theta| \le \pi} \cdots \right]
\end{aligned}
$$

Example 1: Stirling's Formula

$$\frac{1}{n!} = [z^n]e^z = \frac{1}{2\pi i}\oint_{|z|=r} \frac{\exp(z)}{z^{n+1}}\,dz = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\exp(re^{i\theta})}{r^n e^{ni\theta}}\,d\theta$$

$$= \frac{e^r}{2\pi r^n}\Big[\int_{-\delta}^{+\delta} \exp\left((r-n)i\theta - (1/2)r\theta^2 + O(r|\theta|^3)\right) d\theta$$

$$+O(1)\int_{\delta\le|\theta|\le\pi} \exp\left(-cr\theta^2\right) d\theta\Big]$$

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$$|\exp(re^{i\theta})| = \exp(r\cos\theta) = e^r\exp(-r(1-\cos\theta)) \le e^r\exp(-cr\theta^2)$$

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$$r = n \qquad r\delta^2 \to \infty \qquad r\delta^3 \to 0$$

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$$+O(1)\int_{\delta\le|\theta|\le\pi} \exp\left(-cr\theta^2\right)d\theta\Big]$$

$$= \frac{e^n}{2\pi n^n}\left[n^{-1/2}\left(\sqrt{2\pi}+o(1)\right) + O(e^{-cn\delta^2})\right]$$

W. K. Hayman

A generalisation of Stirling's formula

Journal für die reine und angewandte Mathematik

vol 196 (1956) 67–95.

/***/

Hayman's method can be used to prove

$$p(n) = \frac{1}{4n\sqrt{3}} \exp\left(Cn^{1/2} - (\frac{C}{48} + \frac{1}{C})n^{-1/2} + \cdots \right)$$

($C = \pi\sqrt{2/3}$)

Example 2: Binary Matrices

Let $\mathbf{s} = s_1, s_2, \ldots, s_m$, $\mathbf{t} = t_1, t_2, \ldots, t_n$ be integer sequences.

Assume $\sum s_j = \sum t_k$.

Let $B(m, n; \mathbf{s}, \mathbf{t})$ be the number of $m \times n$ matrices over $\{0, 1\}$ with row sums $\mathbf{s}$ and column sums are $\mathbf{t}$.

Then

$$B(m, n; \mathbf{s}, \mathbf{t}) = [x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}] \prod_{\substack{1 \le j \le m \\ 1 \le k \le n}} (1 + x_j y_k)$$

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$$= \frac{1}{(2\pi)^{m+n}} \oint \cdots \oint \frac{\prod (1 + x_j y_k)}{x_1^{s_1+1} \cdots y_n^{t_n+1}} dx dy$$

The Integral

$$x_j = q_j e^{i\theta_j} \qquad y_k = r_k e^{i\phi_k}$$

$$1 + x_j y_k = (1 + q_j r_k) \left(1 + \frac{q_j r_k}{1 + q_j r_k}(e^{i(\theta_j + \phi_k)} - 1)\right)$$

$$B(m, n; \mathbf{s}, \mathbf{t}) = \frac{\prod_{j=1}^{m} \prod_{k=1}^{n} (1 + q_j r_k)}{(2\pi)^{m+n} \prod_{j=1}^{m} q_j^{s_j} \prod_{k=1}^{n} r_k^{t_k}}$$

$$\times \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j=1}^{m} \prod_{k=1}^{n} (1 + \lambda_{jk}(e^{i(\theta_j + \phi_k)} - 1))}{\exp(i \sum_{j=1}^{m} s_j \theta_j + i \sum_{k=1}^{n} t_k \phi_k)} \, d\theta d\phi$$

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$$\times \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j=1}^{m} \prod_{k=1}^{n} (1 + \lambda_{jk} (e^{i(\theta_j + \phi_k)} - 1))}{\exp(i \sum_{j=1}^{m} s_j \theta_j + i \sum_{k=1}^{n} t_k \phi_k)} \, d\theta d\phi$$

$$\lambda_{jk} = \frac{q_j r_k}{1 + q_j r_k}$$

The Integrand

$$\frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+\lambda_{jk}(e^{i(\theta_j+\phi_k)}-1))}{\exp(i\sum_{j=1}^{m}s_j\theta_j+i\sum_{k=1}^{n}t_k\phi_k)}$$

Get $m+n$ saddlepoint equations

The Integrand

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Get $m+n$ saddlepoint equations

$$\sum_{k=1}^{n}\lambda_{jk} \quad = \quad s_j$$

The Integrand

$$\frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+\lambda_{jk}(e^{i(\theta_j+\phi_k)}-1))}{\exp(i\sum_{j=1}^{m}s_j\theta_j+i\sum_{k=1}^{n}t_k\phi_k)}$$

Get $m+n$ saddlepoint equations

$$\sum_{k=1}^{n}\frac{q_j r_k}{1+q_j r_k} = s_j$$

The Integrand

$$\frac{\prod_{j=1}^{m}\prod_{k=1}^{n}(1+\lambda_{jk}(e^{i(\theta_j+\phi_k)}-1))}{\exp(i\sum_{j=1}^{m}s_j\theta_j+i\sum_{k=1}^{n}t_k\phi_k)}$$

Get $m+n$ saddlepoint equations

$$\sum_{j=1}^{m}\frac{q_j r_k}{1+q_j r_k} \quad = \quad t_k$$

Comparison of the Two Examples

	$1/n!$ FirstExample	$B(m, n; \mathbf{s}, \mathbf{t})$ SecondExample
==========	==========	==========
primary $\int$	$\exp(-(1/2)B\theta^2)$	diagonalize quad. form
beyond quadratic	$O(\|\theta\|^3)$	$c_{jk\ell}\theta_j\theta_k\phi_\ell + d_{jk}\theta_j^2\phi_k$ $+ \cdots + O((\cdots)^5)$
secondary $\int$	$\delta \le \|\theta\| \le \pi$	complicated

Consequence

$$B(m,n;\mathbf{s},\mathbf{t}) = \frac{\prod_j \binom{n}{s_j} \prod_k \binom{m}{t_k}}{\binom{mn}{\lambda mn}}$$

$$\times \quad \exp\left(-\frac{1}{2}\left(1-\frac{R}{2Amn}\right)\left(1-\frac{C}{2Amn}\right)+o(1)\right)$$

Consequence

$$B(m,n;\mathbf{s},\mathbf{t}) = \frac{\prod_j \binom{n}{s_j} \prod_k \binom{m}{t_k}}{\binom{mn}{\lambda mn}}$$

$$\times \quad \exp\left(-\frac{1}{2}\left(1-\frac{R}{2Amn}\right)\left(1-\frac{C}{2Amn}\right)+o(1)\right)$$

$$A = (1/2)\lambda(1-\lambda), \quad \lambda = \text{density}$$

$$R = \sum_{j=1}^{m}(s_j - s)^2, \quad C = \sum_{k=1}^{n}(t_k - t)^2$$

References

Brendan McKay, Catherine Greenhill, & erc

Asymptotic enumeration of dense 0-1 matrices
with specified line sums

Journal of Combinatorial Theory, series A
vol 115 (2008) 32–66.

See also A. Barvinok & J.A. Hartigan `arXiv:0910.2477`
for treatment of non-negative integer matrices

Latin Rectangles

Another two-parameter asymptotic counting problem

How many $k \times n$ Latin rectangles are there ?

Erdos & Kaplansky	1946	$k = O(\log n)^{3/2-\epsilon})$
Yamamoto	1951	$k = o(n^{1/3})$
Stein	1978	$k = o(n^{1/2})$
Godsil & McKay	1990	$k = o(n^{6/7})$

$$(n!)^k \left(\frac{(n)_k}{n^k}\right)^n \left(1 - \frac{k}{n}\right)^{-n/2} e^{-k/2}$$

Main Symbols

$$
\begin{array}{ll}
n & \text{the height of the pHm} \\
t & \text{the width} \\
d & = \binom{n}{2} \text{ the dimension of an integral} \\
\delta & \text{defines primary/secondary regions.}
\end{array}
$$

y and $Z(y)$

Given vector y of height n

$$y = \begin{bmatrix} \vdots \\ y_j \\ \vdots \end{bmatrix} \quad \in \quad \{\pm 1\}^n,$$

define the vector of inner products, $Z(y)$, by

$$Z(y) = \begin{bmatrix} \vdots \\ y_j y_k \\ \vdots \end{bmatrix} \quad \in \quad \{\pm 1\}^d.$$

Example of $Z(y)$

$$
\begin{array}{cccccccc}
+ & + & + & + & - & - & - & - \\
+ & + & - & - & + & + & - & - \\
+ & - & + & - & + & - & + & - \\
- & - & + & + & + & + & - & -
\end{array}
$$

$$
\downarrow
$$

$$
Z
$$

$$
\downarrow
$$

$$
\begin{array}{cccccccc}
+ & + & - & - & - & - & + & + \\
+ & - & + & - & - & + & - & + \\
- & - & + & + & - & - & + & + \\
+ & - & - & + & + & - & - & + \\
- & - & - & - & + & + & + & + \\
- & + & + & - & + & - & - & +
\end{array}
$$

H_{nt} as a constant term

Define $M = M_n = \{Z(y) : y \in \{\pm 1\}^n\}$; $|M| = 2^{n-1}$

$$
\begin{aligned}
H_{nt} &= 2^t \times \#\{(\vec{m}_1, \ldots, \vec{m}_t) : \vec{m}_k \in M \ \& \ \textstyle\sum_k \vec{m}_k = \vec{0}\} \\
&= 2^t \times [x_{12}^0 \cdots x_{n-1n}^0] \left(\textstyle\sum_{\vec{m} \in M} \prod_{jk} x_{jk}^{m_{jk}} \right)^t
\end{aligned}
$$

Example of Previous Formula

$H_{38} \quad = \quad 2^8 \times$ the constant term in the expansion of

$$\left(uvw + \frac{v}{uw} + \frac{u}{vw} + \frac{w}{uv}\right)^8$$

corresponding to

$$y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$Z(y) = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Constant Term as Integral

Let $x_{jk} = e^{i\lambda_{jk}}$ and define

$$\psi(\lambda) = \frac{1}{|M|} \sum_{\vec{m} \in M} e^{i\lambda \cdot \vec{m}}.$$

Then,

$$H_{nt} = 2^t \times [x_{12}^0 \cdots x_{n-1n}^0] \left(\sum_{\vec{m} \in M} \prod_{jk} x_{jk}^{m_{jk}} \right)^t$$

$$= \frac{2^{nt}}{(2\pi)^d} \times \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} \psi(\lambda)^t d\lambda.$$

Primary/Secondary

From now on, $4|t$.

$$H_{nt} = \frac{2^{nt}}{(2\pi)^d} \left[2^{(n-1)^2} \int_{B_\delta} \psi(\lambda)^t d\lambda + \int_{R_\delta} \psi(\lambda)^t d\lambda \right]$$

$$B_\delta = \{\lambda : |\lambda| \le \delta\}$$

$$R_\delta = \text{ the } R(\text{est}) \text{ of } [-\pi, +\pi]^d.$$

Why the factor $2^{(n-1)^2}$?

The Primary Region

$$\psi(\lambda) = \frac{1}{|M|} \sum_{\vec{m} \in M} e^{i\lambda \cdot \vec{m}}$$

Where does $|\psi(\lambda)| = 1$?

Start with $\lambda = \vec{0}$; place π in $2^{\binom{n}{2}}$ ways;
add $\pi/2$ in $2^{\binom{n-1}{2}}$ ways.

$$\binom{n}{2} + \binom{n-1}{2} = (n-1)^2$$

Without $4|t$, the factor $(1^t + (-1)^t + (i)^t + (-i)^t)$ vanishes.

Taylor Series for Primary

$$\psi(\lambda) = \frac{1}{|M|} \sum_{\vec{m} \in M} e^{i\lambda \cdot \vec{m}}$$

$$\psi(\lambda) = \sum_{G \in \mathcal{M}_{\text{even}}(n)} \prod_{jk} \frac{\lambda_{jk}^{\mu_{jk}(G)}}{\mu_{jk}(G)!},$$

$\mathcal{M}_{\text{even}}(n)$ equals even multigraphs on $[n] = \{1, 2, \ldots, n\}$

Series is convergent for all λ, but we would like

$$\psi(\lambda) = \exp(\cdots).$$

Primary Integral

Assuming $n\delta \to 0$,

$$\psi(\lambda)^t = \exp\left(-(t/2) \parallel \lambda \parallel^2 + O(tn^3\delta^3)\right)$$

Assuming $t\delta^2 \to \infty$,

$$\int_{-\delta}^{+\delta} \exp\left(-(t/2)x^2\right) \quad = \quad \sqrt{\frac{2\pi}{t}}\left(1 + O(e^{-t\delta^2/2})\right).$$

Provided $tn^3\delta^3 \to 0$,

$$\int_{B_\delta} \psi(\lambda)^t = \left(\frac{2\pi}{t}\right)^{d/2}(1 + o(1)).$$

Secondary Integral

de Launey and Levin prove that for any k, $1 \le k \le n$,

$$|\psi(\lambda)|^2 \;\le\; \frac{1}{2} + \frac{1}{2}\prod_{\substack{j=1\\ j\neq k}}^{n} \cos 2\lambda_{jk}$$

This gives

$$\left|\int_{R_\delta} \psi(\lambda)^t\right| \;\le\; (2\pi)^d\, e^{-ct\delta^2}$$

Combining the Two

$$H_{nt} \;=\; \frac{2^{nt}}{(2\pi)^d}\left[2^{(n-1)^2}\int_{B_\delta}\psi(\lambda)^t d\lambda \;+\; \int_{R_\delta}\psi(\lambda)^t d\lambda\right]$$

$$H_{nt} \;=\; \frac{2^{nt}}{(2\pi)^d}\left[2^{(n-1)^2}\left(\frac{2\pi}{t}\right)^{d/2}(1+o(1)) \;+\; O(1)(2\pi)^d\, e^{-ct\delta^2}\right]$$

$$H_{nt} \;=\; 2^{nt}\left[2^{(n-1)^2}(2\pi t)^{-d/2}(1+o(1)) \;+\; O(1)\, e^{-ct\delta^2}\right]$$

The assumptions: $t\delta^2 \to \infty$, $tn^3\delta^3 \to 0$.

Conditions on δ

$$H_{nt} \;=\; 2^{nt}\left[2^{(n-1)^2}(2\pi t)^{-d/2}(1+o(1)) \;+\; O(1)e^{-ct\delta^2}\right]$$

We arrive at deLauney and Levin's formula,

$$H_{nt} \;\sim\; \frac{2^{nt+(n-1)^2}}{(2\pi t)^{d/2}},$$

obtained under assumption of $\delta > 0$ such that $tn^3\delta^3 \to 0$ and $t\delta^2 = \Omega(n^2 \log t)$.

Setting δ

$$H_{nt} \sim \frac{2^{nt+(n-1)^2}}{(2\pi t)^{d/2}},$$

Necessary conditions: $tn^3\delta^3 \to 0$ and $t\delta^2 = \Omega(n^2 \log t)$.

$$\delta = C\sqrt{\frac{n^2 \log t}{t}}$$

$$t\delta^2 = \Omega(n^2 \log t) \to \infty$$

$$tn^3\delta^3 = O(n^6 \log^{3/2}(t)/t^{1/2}) \to 0$$

Can do for $t \geq n^{12+\epsilon}$.

Revisiting Secondary $\int$

$$|\psi(\lambda)|^2 \;\leq\; \frac{1}{2} + \frac{1}{2}\prod_{\substack{j=1\\ j\neq k}}^{n} \cos 2\lambda_{jk}$$

$$|\psi(\lambda)|^2 \;\leq\; e^{-c\delta^2}, \quad \lambda \in R_\delta$$

Worst case: $\delta \leq |\lambda_{jk}| \leq \pi$ for one pair jk

In fact: $\delta \leq |\lambda_{jk}| \leq \pi$ for $\leq cn^2$ pairs $jk \Longrightarrow$ small measure

Leads to

$$\left|\int_{R_\delta} \psi(\lambda)^t d\lambda\right| \;\leq\; (2\pi)^d \, e^{-ctn\delta^2},$$

new factor of n in the exponent

Resetting δ

$$H_{nt} \quad = \quad 2^{nt} \left[\frac{2^{(n-1)^2}}{(2\pi t)^{d/2}} (1 + \cdots) \;+\; O(e^{-ctn\delta^2}) \right]$$

For $tn\delta^2 = \Omega(n^2 \log t)$, secondary is negligible vs primary

$$\delta \quad = \quad C \sqrt{\frac{n \log t}{t}}$$

previously n^2 in the numerator

Back to Primary $\int$

We'll assume $t \geq n^{4+\epsilon}$. (More than needed for $n\delta = o(1)$.)

$$\psi(\lambda)^t = \exp\left(-(t/2) \parallel \lambda \parallel^2 - it \sum_{j<k<\ell} \lambda_{jk}\lambda_{j\ell}\lambda_{k\ell} + O(tn^2 \sum_{jk} \lambda_{jk}^4) \right)$$

Integrate one variable at a time, starting with $x \equiv \lambda_{12}$.

$$\int_{-\delta}^{+\delta} \exp\left(-(t/2)x^2 - itxB + O(tn^2x^4)\right) dx$$

$$B = \sum_{\ell=3}^{n} \lambda_{1\ell}\lambda_{2\ell} = O(n\delta^2) \to 0$$

Primary $\int$, continued

$$\int_{-\delta}^{+\delta} \exp\left(-(t/2)x^2 - itxB + O(tn^2x^4)\right) dx$$

Since $tn^2\delta^4 \approx n^4/t \to 0$,

$$\int_{-\delta}^{+\delta} \exp\left(-(t/2)x^2 - itxB\right) \left(1 + O(tn^2x^4)\right) dx$$

$$\int_{-\infty}^{+\infty} \exp\left(-(t/2)x^2 - itxB\right) \left(1 + O(tn^2x^4)\right) dx + O(e^{-ct\delta^2})$$

Complete Square

$$\int_{-\infty}^{+\infty} \exp\left(-(t/2)x^2 - itxB\right)\,\left(1 + O(tn^2x^4)\right) dx$$

$$e^{-tB^2/2}\int_{-\infty}^{+\infty} \exp\left(-(t/2)x^2\right)\,\left(1 + O(tn^2(x^4 + B^4))\right) dx$$

$$e^{-tB^2/2}\sqrt{\frac{2\pi}{t}}\,\left(1 + O(n^2/t) + O(tn^2B^4)\right)$$

Power-sum bounds for B^2, B^4

$$B = \sum_{\ell=3}^{n} \lambda_{1\ell}\lambda_{2\ell} \leq (1/2) \sum_{\ell=3}^{n} (\lambda_{1\ell}^2 + \lambda_{2\ell}^2)$$

$$B^2 = O(n) \sum_{\ell=3}^{n} (\lambda_{1\ell}^4 + \lambda_{2\ell}^4)$$

$$B^4 = O(n^3) \sum_{\ell=3}^{n} (\lambda_{1\ell}^8 + \lambda_{2\ell}^8)$$

Combining

Previous

$$e^{-tB^2/2}\sqrt{\frac{2\pi}{t}}\,\left(1+O(n^2/t)+O(tn^2B^4)\right)$$

equals

$$\sqrt{\frac{2\pi}{t}}\,\exp\left(O(n^2/t)+O(tB^2)+O(tn^2B^4)\right)$$

Combining, continued

Previous:

$$\sqrt{\frac{2\pi}{t}}\,\exp\left(O(n^2/t)+O(tB^2)+O(tn^2B^4)\right)$$

Thence, with the power-sum bounds for B^2, B^4

$$\text{Primary}\int_{B_\delta}\cdots=\exp\left(O(n^2/t)\right)\,\sqrt{\frac{2\pi}{t}}$$

$$\times\ \exp\left(O(tn)\sum_{\ell=3}^{n}(\lambda_{1\ell}^4+\lambda_{2\ell}^4)+O(tn^5)\sum_{\ell=3}^{n}(\lambda_{1\ell}^8+\lambda_{2\ell}^8)\right)$$

$$\times\ \text{integrand with remaining } d-1 \text{ variables}$$

Later Variables

When integrating with respect to $x \equiv \lambda_{jk}$,

$$\int_{-\delta}^{+\delta} \exp\left(-(t/2)x^2 - itxB_{jk} + O(tn^2x^4) + O(tn^6x^8)\right) dx,$$

because $\lambda_{jk}^4, \lambda_{jk}^8$ terms were inserted $O(n)$ times.
Inductively, after τ variables,

$$\text{Primary} \int_{B_\delta} \cdots = \exp\left(\tau O(n^2/t) + \tau O(n^6/t^3)\right) \left(\frac{2\pi}{t}\right)^{\tau/2}$$

$$\times \ \exp\left(O(tn) \sum_{\ell=k+1}^{n} (\lambda_{j\ell}^4 + \lambda_{k\ell}^4) + O(tn^5) \sum_{\ell=k+1}^{n} (\lambda_{j\ell}^8 + \lambda_{k\ell}^8) \right)$$

$$\times \ \text{integrand with remaining } d - \tau \text{ variables}$$

Integration Complete

When $\tau = d$,

$$\text{Primary} \int_{B_\delta} \cdots = \exp\left(dO(n^2/t) + dO(n^6/t^3)\right) \left(\frac{2\pi}{t}\right)^{d/2}$$

Since, $n^4/t, n^8/t^3 \to 0$, de Launey - Levin formula holds for $t \geq n^{4+\epsilon}$.

Challenge

Want $H_{nt} \sim \cdots$ for t as small as can be

Would at least like to know the limits of “standard methods”

Current δ gives $n\delta = o(1)$ for $t \geq n^{3+\epsilon}$; then must integrate

$$\psi(\lambda)^t = \exp\Big(-(t/2) \parallel \lambda \parallel^2 - it \sum_{j<k<\ell} \lambda_{jk}\lambda_{j\ell}\lambda_{k\ell} + \cdots$$

$$+ \cdots + O(tn^4 \sum_{jk} \lambda_{jk}^6) \Big)$$