The Size of the Largest Antichain in the Partition Lattice

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Abstract

Consider the poset Π_n of partitions of an n-element set, ordered by refinement. The sizes of the various ranks within this poset are the Stirling numbers of the second kind. Let $a = \frac{1}{2} - e\log(2)/4$. We prove the following upper bound for the ratio of the size of the largest antichain to the size of the largest rank:

$$\frac{d(\Pi_n, \leq)}{S(n, K_n)} \quad \leq \quad c_2\, n^a\, (\log n)^{-a-1/4},$$

for suitable constant c_2 and $n > 1$. This upper bound exceeds the best known lower bound for the latter ratio by a multiplicative factor of $O(1)$.

1. Statement of the Theorem

A *partition* of the set $[n] = \{1, 2, \ldots n\}$ is a collection of nonempty, pairwise disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. Let Π_n denote the set of partitions of $[n]$. Given two partitions $x, y \in \Pi_n$, we say that x *refines* y if x can be obtained from y by further partitioning one or more blocks of y. The reflexive closure of this relation, denoted $x \leq y$, is reflexive, transitive, and antisymmetric; hence, the pair $(\Pi_n, \leq)$ is a finite partially ordered set, or *poset*.

An *antichain* in a poset $(P, \leq)$ is a set of elements pairwise incomparable with respect to the order relation, and $d(P, \leq)$ is the maximum size of any antichain in the poset $(P, \leq)$. In the poset $(\Pi_n, \leq)$ those partitions having a specified number k of blocks form an antichain whose size is $S(n, k)$, the Stirling number of the second kind. For fixed n the sequence $S(n, k)$ is unimodal in k, and it has been of interest to investigate the relationship between $d(\Pi_n, \leq)$ and the largest Stirling number $S(n, K_n)$. Defining the constant a by (all logarithms in this paper are natural)

$$a \quad = \quad \frac{2 - e\log 2}{4},$$

we may state our Theorem.

Theorem. Let $(\Pi_n, \leq)$ be the poset of partitions of $[n]$ ordered by refinement, and let $S(n, K_n)$ be $\max_k S(n, k)$, the largest Stirling number of the second kind. Then for suitable constants c_1, c_2, and $n > 1$,

$$c_1\, n^a\, (\log n)^{-a-1/4} \quad \leq \quad \frac{d(\Pi_n, \leq)}{S(n, K_n)} \quad \leq \quad c_2\, n^a\, (\log n)^{-a-1/4},$$

where $d(\Pi_n, \leq)$ is the size of the largest antichain in the poset$(\Pi_n, \leq)$.

The lower bound in this theorem is proven in [6]; this paper is devoted to the upper bound. We use both | | and # to denote the cardinality of a set, reserving the second for sets defined with curly braces { }. The symbols $c_1, c_2, \ldots c_5, \delta_0, \epsilon_0$ denote positive real constants; it would be possible but distracting to replace these by appropriate explicit values.

2. Background

Determining the largest antichain in a poset originated with the subsets of $[n]$ ordered by inclusion: how big can a collection of subsets be, no two of which are related by inclusion? There is a candidate for the solution: all subsets of a specified size k. The k that gives the best solution of this form is $k = \lfloor n/2 \rfloor$, and it is a theorem of Sperner [21] that no larger collection is possible.

In recent times, any ranked poset whose largest antichain is no bigger than its maximum level is said to possess the *Sperner property.* (Some terminology: Element y is said to *cover* element x in a poset when $x < y$ and there is no z with $x < z < y$; the poset is *ranked* if the rank function $\mathrm{r}(x) = 0$ for x minimal, $\mathrm{r}(y) = \mathrm{r}(x) + 1$ when y covers x, is well defined. Equivalently, a poset is ranked when all maximal chains ending at a prescribed element x have the same length. The subset of all x such that $\mathrm{r}(x) = k$ is called the k-th *level* of the ranked poset; each level is an antichain.)

Many interesting posets arise in combinatorics, supplying researchers with a wealth of problems: identify the maximum level, and determine if the poset is Sperner. Engel's treatise [11] contains 474 references on Sperner theory. The main thrust of this research has been to develop tools for establishing the Sperner property; there are considerably fewer techniques for bounding $d(P, \leq)$ in non-Sperner posets.

After being publicized by Rota [19], the question of whether the partition lattice is Sperner received attention, but defied analysis for a while. Study of the problem inspired many good research papers, some of which led to substantial later work, for instance [9], [12], [15], and [17]. In [5] and [20] two different examples are constructed to prove the strict inequality $d(\Pi_n, \leq)/S(n, K_n) > 1$ for all large n, but in both of these examples the latter ratio is only $1 + o(1)$. The possibility of fitting Π_n into an elegant theory whereby all geometric lattices with some natural property are Sperner was eliminated. Nevertheless, partitions continued to charm a small group of researchers, and the intriguing possibility $d(\Pi_n, \leq) \sim S(n, K_n)$ remained open.

In 1985 Harper gave a heuristic argument [13] that an antichain in Π_n could be 1.69 times as big as $S(n, K_n)$. In later collaborative work [6] we construct an antichain A for which the ratio becomes infinite:

$$c_1\, n^a\, (\log n)^{-a-1/4}\; S(n, K_n) \quad \leq \quad |A|.$$

(This result sometimes appeared in a weaker but easier-to-remember form, with the constant $a = 1/2 - e \log 2/4$ replaced by the slightly smaller $1/35$, and c_1 and the logarithms omitted from the lower bound.) A simpler explanation of our lower bound, given in [7], is this: for any real-valued function $Z : \Pi_n \to R^+$ satisfying

$$x < y \Rightarrow Z(x) + 1 \leq Z(y), \tag{2.1}$$

the set $\{x \in \Pi_n : \lambda - 1/2 < Z(x) \leq \lambda + 1/2\}$ is an antichain. Using Chebyshev's inequality one shows that for some λ the latter set will be at least as large as $cB_n/\sigma(Z)$, where $B_n = |\Pi_n|$ is the n-th Bell number, and $\sigma(Z)$ is the standard deviation of Z as a random variable on Π_n with the uniform probability measure. This general idea is found in early work of Alekseev [2]. Engel [10] developed the idea further, and for a general poset defined a function Z satisfying (2.1) and having minimum variance to be an *optimal representation.*

Meanwhile, with all the effort to find large antichains, no upper bound on $d(\Pi_n, \leq)$ which is $o(B_n)$ has been previously announced. Given the *ad hoc* nature of the function Z in [6], it is surprising to learn that the lower bound found there is correct within multiplication by $O(1)$. Equally surprising is the route by which the upper bound is obtained. Namely, we will define a new partial order on the set of partitions by discarding almost all of the relations in the old order. Being obtained by this sort of a relaxation of $(\Pi_n, \leq)$, the biggest antichain in this new sparser order is at least as big as $d(\Pi_n, \leq)$. The new order is presented in the next section.

3. The New Order

Since we are interested in two different order relations on the set Π_n of partitions of $[n]$, we shall use the notations

$$(\Pi_n, \leq) \quad \text{and} \quad (\Pi_n, \preceq_{\text{no}})$$

for the two different posets. The relation $\leq$ is the usual refinement relation, and the relation $\preceq_{\text{no}}$ is a new order which we are about to define. Both $d(\Pi_n, \leq)$ and $d(\Pi_n, \preceq_{\text{no}})$ refer to largest possible antichains of partitions, but with respect to different order relations.

Let h be a positive integer; given two partitions $x, y \in \Pi_n$, we say that x *h-refines* y if x can be obtained from y by partitioning one or more blocks of y of size $2h$ into two blocks of size h. The reflexive closure of this relation, denoted $x \preceq_{\text{no}} y$, is reflexive, transitive, and antisymmetric; the pair $(\Pi_n, \preceq_{\text{no}})$ is a ranked poset.

What is the rank function of this new order? Define $N_i(x)$ to be the number of blocks of size i in the partition x. The function N_{2h} is the rank function for $(\Pi_n, \preceq_{\text{no}})$. For comparison, we note that the rank function for the refinement order is $\mathrm{r}(x) = n - |x|$, n minus the number of blocks of x. A partition x can have different ranks in these two orders. For example, if $n \neq 2h$, the unique partition in $(\Pi_n, \leq)$ of rank $n-1$ has rank 0 in $(\Pi_n, \preceq_{\text{no}})$. Since $x \preceq_{\text{no}} y$ specifies that x be a refinement of y of a special sort, we see the implication

$$x \preceq_{\text{no}} y \Rightarrow x \leq y.$$

Consequently, any set which is an antichain in $(\Pi_n, \leq)$ is also an antichain in $(\Pi_n, \preceq_{\mathrm{no}})$, and we have the important relation

$$d(\Pi_n, \leq) \quad \leq \quad d(\Pi_n, \preceq_{\mathrm{no}}).$$

We will need a bound on the number of partitions having unusually high or low rank in the poset $(\Pi_n, \preceq_{\mathrm{no}})$, and that is the purpose of the lemma in the next section.

4. Some estimates for the level sets of N_h

Given the preceding section's discussion of the new order, one anticipates the need for information about the distribution of N_h and N_{2h}. In this section we obtain the necessary result. Starting now, and continuing throughout the rest of the paper, we will need the parameter r, $(\ = r(n))$, defined by the equation

$$re^r \quad = \quad n.$$

It is clear that r lies between $\log n - \log\log n$ and $\log n$, and so $r \sim \log n$ as $n \to \infty$.

Later we will define h, as a function of n, to be a specific integer, but for now we state and prove results which apply uniformly to all h in a range. Let $N_{h,m,n}$ equal the number of partitions of $[n]$ having exactly m blocks of size h:

$$N_{h,m,n} \quad = \quad \#\{x \in \Pi_n : N_h(x) = m\},$$

and let X, $(= X(n,h))$, be defined by

$$X \quad = \quad r^h/h!.$$

Lemma 1. For all n sufficiently large and all integers h satisfying $4 \leq h \leq 2r$, we have

$$\sum_{|m-X|>2(rX)^{1/2}} N_{h,m,n} \quad < \quad n^{-1}B_n.$$

Before proving lemma 1, we need three preliminary results which we refer to as "sub lemmas." The conditioning device used here to prove Lemma 1 is found in [3] and discussed in [18].

For an n-vector $\vec{\gamma} = (\gamma_1, \ldots \gamma_n)$, we use $\mathrm{s}(\vec{\gamma})$ to denote the sum $\gamma_1 + 2\gamma_2 + \cdots n\gamma_n$. Let us first recall the classical combinatorial formula [8, Thm B, p. 205] for the number of partitions x having a prescribed *shape* vector $\vec{N}(x) = (N_1(x), \ldots N_n(x))$:

$$\#\{x \in \Pi_n : \vec{N}(x) = \vec{\gamma}\} \quad = \quad \frac{n!}{\prod_j (j!)^{\gamma_j}\gamma_j!}, \quad \mathrm{s}(\vec{\gamma}) = n. \tag{4.1}$$

Of course, summing the quantity on the right of the equality over all n-tuples $\vec{\gamma}$ for which $\mathrm{s}(\vec{\gamma}) = n$ yields the Bell number B_n. Consider a sequence of independent random variables $\vec{Y} = (Y_1, \dots Y_n)$ in which Y_j is Poisson distributed with parameter $r^j/j!$.

Sub lemma a. Let $\vec{g} = (g_1, \dots g_n)$ be an n-tuple of integers satisfying $\mathrm{s}(\vec{g}) = n$. Then the probability that $\vec{N}(x)$ equals $\vec{g}$ when partition x is selected uniformly at random from Π_n is the same as the probability that $\vec{Y} = \vec{g}$ *conditioned* on the event that $\mathrm{s}(\vec{Y}) = n$.

Proof. Direct substitution into the Poisson distribution function, (4.1), and the remark immediately following (4.1) reveal

$$\begin{aligned} \Pr\big(\mathrm{s}(\vec{Y}) = n\big) &= \sum_{\mathrm{s}(\vec{\gamma})=n} \prod_{j=1}^{n} e^{-r^j/j!} (r^j/j!)^{\gamma_j}/\gamma_j! \\ &= \frac{r^n}{n!} e^{-R} B_n, \quad R = \sum_{j=1}^{n} r^j/j!, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \Pr\big(\mathrm{s}(\vec{Y}) = n \ \&\ \vec{Y} = \vec{g}\big) &= \Pr(\vec{Y} = \vec{g}) \\ &= \prod_{j=1}^{n} e^{-r^j/j!} (r^j/j!)^{g_j}/g_j! \\ &= \frac{r^n}{n!} e^{-R} \ \#\{x \in \Pi_n : \vec{N}(x) = \vec{g}\}. \end{aligned}$$

Thus,

$$\Pr\big(\vec{Y} = \vec{g} \mid \mathrm{s}(\vec{Y}) = n\big) = \frac{\Pr(\mathrm{s}(\vec{Y}) = n \ \&\ \vec{Y} = \vec{g})}{\Pr(\mathrm{s}(\vec{Y}) = n)} = \frac{\#\{x \in \Pi_n : \vec{N}(x) = \vec{g}\}}{B_n},$$

and the proof is complete.

Sub lemma a. gives us a way to transfer knowledge about the familiar Poisson variable Y_h into knowledge about the distribution of N_h. To complete the transfer, we need a lower bound on the probability of the conditioning event.

Sub lemma b. For suitable constant $c_3 > 0$ and all sufficiently large n,

$$\Pr\big(\mathrm{s}(\vec{Y}) = n\big) > c_3 e^{-r/2}/r.$$

Proof. Since

$$R = \sum_{j=1}^{n} r^j/j! = e^r - 1 + o(1),$$

the assertion of the lemma is immediate from the Moser-Wyman [16,(3.3) and (3.4)] asymptotic formula

$$B_n \;=\; \frac{n!}{r^n}\,\frac{\exp\{e^r-1\}}{\sqrt{2\pi}\,re^{r/2}}\,\big(1+o(1)\big)$$

and (4.2).

The third sub lemma is a simple bound for the tails of the Poisson variable Y_h.

Sub lemma c. For each real number C there exists n_0 such that for $n \geq n_0$ and $4 \leq h \leq 2r$ we have

$$\Pr(|Y_h - X| > C(rX)^{1/2}) \;\leq\; e^{-C^2r/2}.$$

Proof. Let C be given. By the hypotheses for h and Stirling's formula applied to $r^h/h!$ we have

$$r^{3/2}/X^{1/2} \;\to\; 0, \qquad \text{as } n \to \infty. \tag{4.3}$$

Let j_0 be the smallest integer exceeding $X + C(rX)^{1/2}$. The sum $\sum_{j=j_0}^{\infty} e^{-X}X^j/j!$ can be bounded by a geometric series whose ratio is X/j_0. We find

$$\frac{1}{1-X/j_0} \;<\; \frac{1+C(r/X)^{1/2}}{C(r/X)^{1/2}} \;\leq\; \frac{2X^{1/2}}{Cr^{1/2}}, \tag{4.4}$$

provided $C(r/X)^{1/2} \leq 1$, which is true, by (4.3), for large n. The first term of the sum can be bounded using Stirling's formula and routine estimates:

$$\begin{aligned}
e^{-X}X^{j_0}/j_0! \;&=\; X^{-1/2}\exp\big\{-X \;+\; j_0 \;+\; j_0\log(X/j_0) \;+\; O(1)\big\}\\
&\leq\; X^{-1/2}\exp\big\{-X \;+\; X \;+\; C(rX)^{1/2} \;+\; 1 \;+\; O(1) \;+\\
&\qquad +\; \big(X + C(rX)^{1/2}\big)\big(-\frac{Cr^{1/2}}{X^{1/2}} \;+\; \frac{\frac{1}{2}C^2r}{X}\big)\big\}\\
&=\; X^{-1/2}\exp\big\{-\tfrac{1}{2}C^2r \;+\; O(1)\big\}.
\end{aligned} \tag{4.5}$$

The latter two relations here hold for large n by condition (4.3). Relations (4.4) and (4.5) together show that $\sum_{j\geq j_0} e^{-X}X^j/j!$ is $o(1)e^{-C^2r/2}$, since C is fixed and $r \to \infty$. The sum for $j \leq j_1$, where j_1 is the largest integer less than $X - C(rX)^{1/2}$, can be treated similarly.

The three sub lemmas combine to prove Lemma 1.

Proof of Lemma 1. Take the real number C in sub lemma c. to be 2. For all n sufficiently large we have

$$\begin{aligned}
\#\{x : |N_h(x) - X| > 2(rX)^{1/2}\}\; B_n^{-1} \;&=\; \Pr\big(|Y_h - X| > 2(rX)^{1/2} \;\big|\; \mathrm{s}(\vec{Y}) = n\big)\\
&\leq\; \frac{\Pr(|Y_h - X| > 2(rX)^{1/2})}{\Pr(\mathrm{s}(\vec{Y}) = n)}\\
&\leq\; c_3^{-1}re^{-3r/2}.
\end{aligned}$$

In the preceding, sub lemma a. gives the first equality, sub lemmas b. and c. imply the last inequality, and the middle inequality is just the obvious $\Pr(E_1 \cap E_2) \leq \Pr(E_1)$. Since $r \sim \log n$, $c_3^{-1} r e^{-3r/2} < n^{-1}$ for all n sufficiently large, and the proof is complete.

5. Some Sperner posets, and their largest ranks

Let μ and h be positive integers, and $P_\mu^{(h)} \subseteq \Pi_{\mu h}$ be the set of partitions of $[\mu h]$ whose block sizes are all h or $2h$. Restricting the refinement relation, we obtain a ranked poset $(P_\mu^{(h)}, \leq)$ whose rank function is N_{2h}. Notice that each partition $x \in P_\mu^{(h)}$ having rank t is related to the same number, $\binom{\mu-2t}{2}$, of partitions y of rank $t+1$; and that each partition $y \in P_\mu^{(h)}$ having rank $t+1$ is related to the same number, $(t+1)\binom{2h}{h}/2$, of partitions x of rank t. Posets whose covering relations have this constant degree property are called *regular*, and such posets are Sperner.

Lemma 2. The posets $P_\mu^{(h)}$ possess the Sperner property; that is, the largest antichain in these posets is equal in size to the largest rank.

Proof. See [1, Prop. 8.43] or [4].

Suppressing for brevity the dependence on μ and h, let R_t be the number of partitions in $P_\mu^{(h)}$ having rank t. The above discussion of regularity shows that

$$R_t \binom{\mu - 2t}{2} \quad = \quad R_{t+1} \, (t+1)\binom{2h}{h}/2. \tag{5.1}$$

Because the ratio $(\mu-2t)(\mu-2t-1)/(t+1)\binom{2h}{h}$ is a decreasing function of t, the numbers R_t, $0 \leq t \leq \mu/2$, are *strictly log concave*:

$$(R_t)^2 \quad > \quad R_{t-1}R_{t+1}.$$

It follows that the sequence R_t increases with t to either a unique, or two equal and consecutive, peak(s), and then decreases [8, Thm A, p. 270]. From now on we let t, $\left(= t(\mu, h) \right)$, denote the location of the leftmost peak. (By "leftmost" we mean that $R_{t-1} < R_t \geq R_{t+1}$). Thus the integer t is defined by the inequalities

$$\frac{(\mu-2t)(\mu-2t-1)}{(t+1)\binom{2h}{h}} \quad \leq \quad 1 \quad < \quad \frac{(\mu-2t+2)(\mu-2t+1)}{t\binom{2h}{h}}, \qquad t \leq \mu/2, \tag{5.2}$$

and we record

$$R_t \quad = \quad \max_{0 \leq \tau \leq \mu/2} R_\tau.$$

If μ and $\binom{2h}{h}$ are large and the same magnitude, then t is the same magnitude, and the ratio $R_t/|P_\mu^{(h)}|$ is the reciprocal of the square root of said magnitude. The precise formulation of this is the next Lemma.

Lemma 3. For every $\delta > 0$ there exists $\epsilon > 0$ such that for all sufficiently large M we have the implication

$$\frac{\mu}{M}, \frac{\binom{2h}{h}}{M} \quad \in \quad [\delta, \delta^{-1}] \quad \Rightarrow \quad \frac{t}{M}, \frac{|P_\mu^{(h)}|}{M^{1/2} R_t} \quad \in \quad [\epsilon, \epsilon^{-1}].$$

Proof. Assume $\delta \leq 1$, else the implication is trivially true. Let k_1, k_2 be the ratios $\mu/M, \binom{2h}{h}/M$ respectively, and let k_3 be the unique solution to

$$\frac{(k_1 - 2k_3)^2}{k_2 k_3} \quad = \quad 1, \qquad 0 < k_3 < k_1/2.$$

We will first show that

$$t \quad = \quad k_3 M \ + \ \theta, \qquad |\theta| < 1. \tag{5.3}$$

Consider the function

$$f(s) \quad = \quad \frac{(\mu - 2s + 2)(\mu - 2s + 1)}{s\binom{2h}{h}}.$$

There is a unique $s_0 < (\mu + 1)/2$ such that $f(s_0) = 1$ because f is decreasing in the latter range. By the definition (5.2) of t, we have

$$t + 1 \quad = \quad \lceil s_0 \rceil. \tag{5.4}$$

Note that $\tau \stackrel{\text{def}}{=} k_3 M$ satisfies the equation

$$\frac{(\mu - 2\tau)^2}{\tau\binom{2h}{h}} \quad = \quad 1.$$

Since

$$\frac{(\mu - 2\tau + 2)(\mu - 2\tau + 1)}{\tau\binom{2h}{h}} \quad > \quad \frac{(\mu - 2\tau)^2}{\tau\binom{2h}{h}} \quad = \quad 1,$$

it follows that $\tau < s_0$. On the other hand, since

$$\frac{(\mu - 2\tau)(\mu - 2\tau - 1)}{(\tau + 1)\binom{2h}{h}} \quad < \quad \frac{(\mu - 2\tau)^2}{\tau\binom{2h}{h}} \quad = \quad 1,$$

it follows that $\tau + 1 > s_0$. Thus $\tau < s_0 < \tau + 1$, from which it follows first that $\tau < \lceil s_0 \rceil < \tau + 2$, and then, by (5.4), that $\tau - 1 < t < \tau + 1$, which is (5.3).

Next, we claim that

$$\delta^3/5 \quad \le \quad k_3 \quad \le \quad (\tfrac{1}{2} - \delta/5)k_1. \tag{5.5}$$

To prove the first inequality, let $s < \delta^3/5$. Then $s < \delta/4$ and $k_2 s < \delta^{-1}\delta^3/5 = \delta^2/5$. Hence,

$$\frac{(k_1 - 2s)^2}{k_2 s} \quad > \quad \frac{(\delta - \delta/2)^2}{\delta^2/5} \quad = \quad 5/4 \quad > \quad 1,$$

which means that k_3 cannot be smaller than $\delta^3/5$.

To prove the second inequality in (5.5), let $\frac{1}{2}k_1 > s > (\frac{1}{2} - \delta/5)k_1$. Multiplying by -2 and adding k_1, it follows that

$$0 \quad < \quad k_1 - 2s \quad < \quad 2\delta k_1/5.$$

Since $\delta \le 1$, we have also $s > 3k_1/10$, and so

$$\frac{(k_1 - 2s)^2}{k_2 s} \quad < \quad \frac{4\delta^2 k_1^2/25}{3\delta k_1/10} \quad = \quad \frac{8}{15}\delta k_1 \quad \le \quad \frac{8}{15} \quad < \quad 1,$$

which means that k_3 cannot be larger than $(\frac{1}{2} - \delta/5)k_1$. This completes the proof of (5.5).

Using (5.3) and (5.5), we can choose ϵ, sufficiently small, depending on δ, such that for M large not only do we have $t/M \in [\epsilon, \epsilon^{-1}]$, but also $\beta/M \in [\epsilon, \epsilon^{-1}]$ where

$$\beta \quad \stackrel{\text{def}}{=} \quad \frac{t(\mu - 2t)}{\mu + 2t}.$$

Also, for future use, we require with no loss that

$$\epsilon \quad < \quad (4\pi)^{-1}. \tag{5.6}$$

Iterating relation (5.1), for $j \ge 0$,

$$R_{t+j} \quad = \quad \frac{(\mu - 2t)_{2j}}{(t+j)_j \binom{2h}{h}^j}\, R_t$$

and

$$R_{t-j} \quad = \quad \frac{(t)_j \binom{2h}{h}^j}{(\mu - 2t + 2j)_{2j}}\, R_t.$$

It is not hard to show, uniformly for $0 \le j \le B^{3/5}$ as $B \to \infty$, that

$$\begin{aligned} (B)_j \quad &= \quad B^j \exp\{-\frac{j^2}{2B} \;+\; O(B^{-1/5})\} \\ (B+j)_j \quad &= \quad B^j \exp\{\frac{j^2}{2B} \;+\; O(B^{-1/5})\}. \end{aligned}$$

Using the latter two estimates in the above formulas for R_{t+j} and R_{t-j}, as well as

$$\frac{(\mu-2t)^2}{t\binom{2h}{h}} \quad = \quad \frac{(k_1-2k_3)^2}{k_3k_2} \times \bigl(1+O(M^{-1})\bigr) \quad = \quad 1+O(M^{-1}),$$

we find

$$R_{t+j} \quad = \quad R_t \exp\bigl\{-\frac{j^2}{2\beta} \; + \; O(M^{-1/5})\bigr\}, \qquad \text{uniformly for } |j| \le M^{3/5}.$$

Since

$$\sum_{|j|\le M^{3/5}} e^{-j^2/2\beta} \quad = \quad \sqrt{2\pi\beta}\bigl(1+o(1)\bigr),$$

we have

$$\sum_{|j|\le M^{3/5}} R_{t+j} \quad = \quad R_t\sqrt{2\pi\beta}\bigl(1+o(1)\bigr).$$

Using the log concavity of R_t, we can bound the sum of R_{t+j} over $|j| > M^{3/5}$ by a pair of geometric series, and the result is negligible in comparison to the sum over $|j| \le M^{3/5}$. So, altogether

$$|P_\mu^{(h)}| \quad = \quad \sum_j R_{t+j} \quad = \quad R_t\sqrt{2\pi\beta}\bigl(1+o(1)\bigr).$$

Now, we have already chosen ϵ so that both t/M and β/M lie in the interval $[\epsilon, \epsilon^{-1}]$. Invoking the condition (5.6) for the final containment in

$$\frac{|P_\mu^{(h)}|}{R_tM^{1/2}} \quad = \quad \sqrt{2\pi}\,\bigl(1+o(1)\bigr)\,\bigl(\beta/M\bigr)^{1/2} \quad \in \quad \sqrt{2\pi}\,\bigl(1+o(1)\bigr)\,[\epsilon^{1/2}, \epsilon^{-1/2}] \quad \subseteq \quad [\epsilon, \epsilon^{-1}],$$

we have proven the lemma.

This concludes our discussion of the posets $P_\mu^{(h)}$, and no further preparatory lemmas are needed. In the next section we complete the proof of the Theorem.

6. The proof completed

For each n define parameters r, h, and M by

$$\begin{aligned} re^r \quad &= \quad n \\ h \quad &= \quad \lfloor er/4 \rfloor \\ M \quad &= \quad r^h/h! \; + \; 2r^{2h}/(2h)!. \end{aligned}$$

To avoid repetition, we adopt the convention that all inequalities asserted in this section are implicitly qualified by the phrase "for n sufficiently large." Let us remark that $4 \le h < 2h \le 2r$, so that both h and $2h$ satisfy the hypotheses of Lemma 1. We also remark that $r^h/h!$, $r^{2h}/(2h)!$, $\binom{2h}{h}$, and M are roughly equal; precisely, using Stirling's formula, we have, for suitable constants c_4, c_5:

$$c_4\, 2^{er/2}\, r^{-1/2} \quad < \quad r^h/h!,\ r^{2h}/(2h)!,\ \binom{2h}{h},\ M \quad < \quad c_5\, 2^{er/2}\, r^{-1/2}. \tag{6.1}$$

Let $x, y \in \Pi_n$ be two partitions. How do we decide if $x \preceq_{\mathrm{no}} y$? Every partition has an h-part: the blocks of sizes h and $2h$; and a non-h-part: the other blocks. The partitions x and y are $\preceq_{\mathrm{no}}$-related precisely when their h-parts are related under refinement and their non-h-parts are identical. In other words, the poset $(\Pi_n, \preceq_{\mathrm{no}})$ is isomorphic to a disjoint union of copies of $P_\mu^{(h)}$ for various repeated values of μ. By Lemma 2,

$$d(\Pi_n, \preceq_{\mathrm{no}}) \quad = \quad \sum_{0 \le \mu \le n/h} \binom{n}{\mu h} R_{t(\mu,h)} A_{n-\mu h},$$

where A_k equals the number of partitions of a k-set none of whose blocks has size h or $2h$.

The part of the preceding summation for which $|\mu - M| > 6(rM)^{1/2}$ is, by Lemma 1, smaller than $2n^{-1}B_n$. By (6.1) there is a $\delta_0 > 0$ such that for all μ appearing in the other part of the summation, where $|\mu - M| \le 6(rM)^{1/2}$, we have $\mu/M, \binom{2h}{h}/M \in [\delta_0, \delta_0^{-1}]$. By Lemma 3, for suitable $\epsilon_0 > 0$,

$$d(\Pi_n, \preceq_{\mathrm{no}}) \quad \le \quad \epsilon_0^{-1} \sum_{|\mu - M| \le 6(rM)^{1/2}} \binom{n}{\mu h} \frac{|P_\mu^{(h)}|}{M^{1/2}} A_{n-\mu h} \ + \ 2n^{-1} B_n.$$

On the other hand, by an obvious combinatorial correspondence,

$$B_n \quad = \quad \sum_{0 \le \mu \le n/h} \binom{n}{\mu h} |P_\mu^{(h)}| A_{n-\mu h},$$

and this gives us altogether

$$d(\Pi_n, \preceq_{\mathrm{no}}) \quad \le \quad \left(\frac{\epsilon_0^{-1}}{M^{1/2}} \ + \ 2n^{-1}\right) B_n.$$

Using Harper's approximation [13]

$$B_n \quad = \quad O(n^{1/2}/r)\ S(n, K_n),$$

the lower bound for M in (6.1), and recalling that $r \sim \log n$, we obtain the upper bound stated in the Theorem. As mentioned earlier, the lower bound was proven in [6], and so the proof of the Theorem is complete.

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