JOURNAL OF COMBINATORIAL THEORY, Series B 33, 276–305 (1982)

Counting Bridgeless Graphs

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Communicated by the Editors

Received June 23, 1981

Methods are developed for finding the number of unlabeled bridgeless or 2-line-connected graphs of any order. These methods are based on cycle index sums, but it is shown how to avoid explicit compution with cycle index sums by using suitable inversion techniques. Similar results are obtained for unlabeled bridgeless graphs by numbers of points and lines, and connected graphs by numbers of points and bridges. Corresponding results for labeled graphs are found as corollaries. When lines or bridges are required as enumeration parameters in the labeled case it is also shown how to obtain improved recurrence relations. The latter appear to have no analog for unlabeled graphs.

INTRODUCTION

A graph is *k-line-connected* if and only if the removal of any set of fewer than k lines leaves a connected graph. Thus a graph is 1-line-connected if and only if it is connected. A *bridge* in a graph G is a line whose removal increases the number of connected components of G. Thus a graph is 2-line-connected if and only if it is connected and contains no bridges. Our object is to count the unlabeled 2-line-connected graphs by number of points. In doing so we also count bridgeless graphs. Counting either variety as labeled graphs is easier, and is deduced as a corollary to the unlabeled counting results. It is also shown how to accomplish these enumerations by number of lines or bridges as well as number of points, in principle a straightforward matter.

By *graph* we mean a simple undirected graph with no loops or multiple lines. The basic facts and definitions concerning connectivity are set out in

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0095-8956/82/060276–30$02.00/0

[9, Chaps. 3 and 5]. Recall that a point in a graph G is a *cutpoint* if its removal increases the number of connected components of G, and a graph is *nonseparable* if it is connected and has no cutpoints. Thus nonseparable graphs are the point-deletion analog of 2-line-connected graphs. The *blocks* of a graph are its maximal nonseparable subgraphs. The lines of any graph are partitioned by its blocks. In [17] it was shown how to count unlabeled nonseparable graphs. The idea was to use cycle index sums instead of ordinary generating functions. By lifting to cycle index sums the known methods for counting graphs with given blocks, and also the known methods for counting all graphs, a relation was obtained which could be solved for the cycle index sum (and hence the numbers) of all nonseparable graphs.

Now, a graph is bridgeless if and only if it does not contain the single line K_2 as a block. Thus the method of [17] already offers a method of counting bridgeless graphs. One first finds the cycle index sum for all blocks, subtracts the terms corresponding to K_2, and then counts the graphs having the remainder as blocks. This approach is very indirect, and involves much explicit computation with cycle index sums.

In this paper a more direct method of counting bridgeless graphs is developed. In a sense it is parallel to, rather than building from, the enumeration of nonseparable graphs. A maximal 2-line-connected subgraph of a graph G is called a *lump*. (There seems to be no previously established terminology for this.) Clearly the lumps of G partition the points, and a line is not contained in some lump just if it is a bridge. We show how to find the cycle index sum for all graphs with given lumps. Comparison with the cycle index sum of all graphs gives a relation which can be solved for the cycle index sum of all 2-line-connected graphs. An inversion device is then applied which allows the solution to be obtained without explicit cycle index sum computations. This device is also applicable to the counting of nonseparable graphs and graphs without points of degree 1 or 2.

Section 1 contains a review of the cycle index sum methods which are required later, along with a development of the inversion procedure. In determining the cycle index of all graphs with given lumps, we start in Section 2 with the rooted case. Then in Section 3 the unrooted case is derived by an approach similar to that which Otter [13] perfected for trees. The essential structural fact relied on is the tree-like structure of the lumps within any connected graph. This was implicit in [1]. However, in the latter paper the attempt was made to avoid the rooted case entirely, and the enumeration method arrived at was not successful. In Section 4 the results of the previous sections are applied to obtain a relation between all connected graphs and all connected bridgeless graphs. Inverting this relation gives our enumeration of unlabeled bridgeless graphs. In Section 5 the methods are extended to include lines or bridges as enumeration parameters. In particular it is shown how to count unlabeled graphs or connected graphs by numbers of points and bridges without recourse to explicit computation with cycle index sums. The labeled analogs of all the unlabeled counting results are derived as corollaries in Section 6. It is also shown how to derive more efficient equations when lines or bridges are required, by methods which do not apply in the unlabeled case. In Section 8 some related results and open questions are discussed.

1. Cycle Index Sum Methods

If Γ is a finite group represented (faithfully or not) as permutations on a finite set, then the *cycle index* $Z(\Gamma)$ is the polynomial

$$Z(\Gamma)=\frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}\prod_i x_i^{j(i,\sigma)}.$$

Here $x_1, x_2,\ldots$ are distinct commuting variables and $j(i,\sigma)$ is the number of i-cycles in the disjoint cycle decomposition of σ. For a graph G let $\Gamma(G)$ denote the automorphism group acting on the point set, and let $Z(G)$ denote the cycle index of $\Gamma(G)$. For instance, when G is the graph K_4-e depicted in Fig. 1, we have $Z(K_4-e)=\frac{1}{4}x_1^4+\frac{1}{2}x_1^2x_2+\frac{1}{4}x_2^2$.

For a set S of graphs, let $Z(S)$ denote the cycle index sum

$$Z(S)=\sum_{G\in S} Z(G).$$

This will be used for sets of unlabeled graphs, in which there are no repeats, so that even when S is infinite the cycle index sum $Z(S)$ is a well-defined member of the power series ring $\mathbb{Q}[\![x_1, x_2, x_3,\ldots]\!]$ over the field $\mathbb{Q}$ of rationals. The substitution of x^i for x_i, $i=1, 2, 3,\ldots$, is indicated by the notation $[x_i\leftarrow x^i]$. Thus $Z(K_4-e)[x_i\leftarrow x^i]=\frac{1}{4}x^4+\frac{1}{2}x^4+\frac{1}{4}x^4=x^4$. In general, we say that a monomial $\Pi a_i^{j(i)}$ has *weight* $\sum_i ij(i)$. Then if G has p points and $\sigma\in\Gamma(G)$, every monomial in $Z(G)$ has weight p and so $Z(G)[x_i\leftarrow x^i]=x^p$. For a set S of unlabeled graphs, then, we have

$$S(x)=Z(S)[x_i\leftarrow x^i]$$

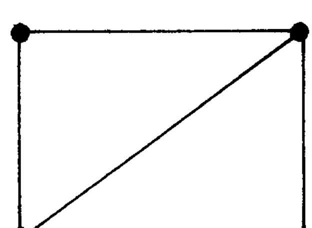

FIG. 1. K_4-e.

for the ordinary generating function in which the coefficient of x^p is the number of graphs in S having exactly p points.

For each result in the classical Pólya theory of enumeration there is a corresponding generalization in which power series are replaced by cycle index sums. Pólya's Hauptsatz rests on the fact that the number of orbits of a finite permutation group is the average over the group of the number of fixed points. This fact is usually called Burnside's lemma [11, p. 39]. Pólya applied Burnside's lemma to functions in Y^X which are inequivalent under the action of a group Γ on X. We suppose that the range elements in Y are assigned nonnegative integer weights, and that their generating function by weight is $F(x)$. Then the generating function by total weight of the Γ-inequivalent functions is obtained from $Z(\Gamma)$ by replacing each variable x_i by $F(x^i)$; see [14 or 11, p. 42]. The result of this operation on $Z(\Gamma)$ and $F(x)$ is denoted $Z(\Gamma)[F(x)]$.

Now if the members of Y are graphs with order as the weight, then $F(x)$ can be replaced by the cycle index sum $Z(Y)$. The replacement $x_i \leftarrow F(x^i)$ generalizes to $x_i \leftarrow Z(Y)[x_j \leftarrow x_{ij}]$, and the latter is abbreviated to $[Z(Y)]$. We can then state the following generalization of Pólya's Hauptsatz, which is [17, Eq. (9)].

COMPOSITION THEOREM. *The cycle index sum of the Γ-inequivalent functions in Y^X is $Z(\Gamma)[Z(Y)]$.*

Here the automorphism group of a function in Y^X is a generalized wreath product of its stabilizer in Γ over the automorphism groups of its values in Y. This turns out to be the natural definition in a number of applications.

The development of [17] was based on [16], in which a weighted form of Burnside's lemma was proved in order to obtain a cycle index sum for superpositions of two sets. The main result is called the composition theorem in the exposition of [11, Chap. 8]. An equivalent to the composition theorem was also proved independently in [3].

Let $\Gamma = \Gamma(G)$ be the automorphism group of a graph G in the composition theorem. Then the cycle index sum for functions from $V(G)$ to Y which are inequivalent under all automorphisms of G is

$$Z(G)[Z(Y)]. \tag{1.1}$$

Now if (1.1) is summed over all graphs G in a set B we find

$$\sum_{G \in B} Z(G)[Z(Y)] = Z(B)[Z(Y)]. \tag{1.2}$$

The equality in (1.2) follows because the map from $\mathbb{Q}[\![x_1, x_2,...]\!] \to \mathbb{Q}[\![x_1, x_2,...]\!]$ obtained by evaluating x_j at $Z(Y)[x_i \leftarrow x_{ij}]$ is a ring homomorphism. In our applications of the composition theorem, expressions of form (1.2) yield equations of the form

$$Z(A) = Z(B)[Z(Y)], \tag{1.3}$$

where A, B, and Y are sets of graphs, and where $Z(A)$ and $Z(Y)$ are known. In this situation we will want to compute $B(x)$ and we will have that $Z(Y) = x_1 + Z_2(Y)$, where $Z_2(Y)$ contains only terms of weight 2 or greater.

From Eq. (1.3) it is possible to recover $Z(B)$ by a simple comparison of coefficients. This yields $B(x)$ by the equation $B(x) = Z(B)[x]$. Computationally, this method of finding $B(x)$ is very clumsy since cycle index sums must be stored during the computation and since we actually compute $Z(B)$ when only $B(x)$ is desired. To avoid these inefficiencies we refine an idea used by Read to enumerate labeled graphs without endpoints [15]. We invert Eq. (1.3) to an equation of the form

$$Z(A)[\mu(x)] = B(x). \tag{1.4}$$

To do so, first solve for the unique generating function $\mu(x)$ of the form $\mu(x) = x + \sum_{n=2}^{\infty} u_n x^n$ which satisfies

$$Z(Y)[\mu(x)] = x. \tag{1.5}$$

Such a generating function exists because $Z(Y)$ decomposes as $Z(Y) = x_1 + Z_2(Y)$. Using this decomposition in (1.5) we have

$$x_1[\mu(x)] = x - Z_2(Y)[\mu(x)].$$

But $x_1[\mu(x)] = \mu(x)$ so we have

$$x + \sum_{n=2} u_n x^n = x - Z_2(Y)[\mu(x)]. \tag{1.6}$$

Now suppose that the first n coefficients $1, u_1, \ldots, u_n$ of $\mu(x)$ are known. Then the coefficient of x^{n+1} in $x - Z_2(Y)[\mu(x)]$ is determined since each term of $Z_2(Y)$ has weight 2 or more. So u_{n+1} is given recursively by Eq. (1.6). Thus we can solve for a unique generating function $\mu(x)$.

Once $\mu(x)$ is known, we compose $\mu(x)$ into both sides of Eq. (1.3). We obtain

$$Z(A)[\mu(x)] = (Z(B)[Z(Y)])[\mu(x)]. \tag{1.7}$$

Define homomorphisms $\varphi_1\colon \mathbb{Q}[\![x_1, x_2, \ldots]\!] \to \mathbb{Q}[\![x_1, x_2, \ldots]\!]$ and $\varphi_2\colon \mathbb{Q}[\![x_1, x_2, \ldots]\!] \to \mathbb{Q}[\![x]\!]$ by

$$\varphi_1(x_j) = Z(Y)[x_i \leftarrow x_{ij}] \qquad \text{and} \qquad \varphi_2(x_j) = \mu(x^j).$$

As composition of homomorphisms is associative we have

$$\varphi_2(\varphi_1(Z(B))) = (\varphi_2\varphi_1)\, Z(B),$$

or equivalently

$$(Z(B)[Z(Y)])[\mu(x)] = Z(B)[Z(Y)[\mu(x)]]. \tag{1.8}$$

Combining (1.8), (1.7), and (1.5) we obtain

$$Z(A)[\mu(x)] = Z(B)[x] = B(x)$$

and this completes the inversion of (1.3) to (1.4). When we compute $B(x)$ from Eq. (1.4) instead of indirectly from Eq. (1.3), we replace the recursive solution of $Z(B)$ from (1.3) by the solution of $\mu(x)$ using (1.6). The recursive solution of $\mu(x)$ is much easier than the recursive solution of $Z(B)$ because all computations are done with ordinary generating functions instead of with cycle index sums. The inversion of an equation of form (1.3) to form (1.4) will play an important role in Sections 4 and 5.

A *rooted graph* G' is a graph in which one vertex (called the *root*) is designated as special. When a rooted graph is drawn, its root will be circled. Underlying each rooted graph G' is a unique unrooted graph $u(G')$ which is obtained from G' by unrooting the root point. If S is a set of graphs, S' denotes the set of all rooted graphs G' with the property that $u(G')$ is in S. The following result due to Ford relates the cycle index sum of the set S' to the cycle index sum of the set S.

FORD'S THEOREM. *Let S be a set of graphs. Then*

$$Z(S') = x_1 \frac{\partial}{\partial x_1} Z(S).$$

Ford's theorem can be proved using the weighted form of Burnside's lemma (see [17, Theorem 1; 11, Sect. 8.5]). As an example of Ford's theorem, let S be the one element set containing the graph $K_4 - e$ of Fig. 1. Earlier we saw that $Z(S) = \frac{1}{4}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{4}x_2^2$. The set S' contains the two graphs in Fig. 2.

FIG. 2. The set S'.

Each graph in S' has cycle index $\frac{1}{2}x_1^4 + \frac{1}{2}x_1^2 x_2$ and so $Z(S') = x_1^4 + x_1^2 x_2 = x_1(\partial/\partial x_1)\, Z(S)$.

2. Rooted Connected Graphs with Given Lumps

Our object is to count rooted, connected graphs with given lumps in terms of the numbers of rooted lumps. First we give a recursive structural characterization of the former which translates into a functional equation. The equation for the unlabeled case is given in terms of cycle index sums, as these are needed in Section 4 when solving for the total number of bridgeless graphs.

Fix a set of lumps M, let K be the set of connected graphs with lumps from the set M and let K' be the set of rooted, connected graphs with lumps from the set M. We examine the structure of a graph G from the set K'. As the lumps of a graph partition its point set, the root point of G must lie in a unique lump which will be called the *rooted lump* of G and will be denoted $r(G)$.

Let u be a point in $r(G)$ and let v be a point adjacent to u but not in $r(G)$. As u and v lie in distinct lumps, there exists a line e whose removal leaves u and v in different connected components of $G - e$. Obviously, that line must be $\{u, v\}$ and so points in $r(G)$ are joined to points not in $r(G)$ by bridges.

Consider the connected component of $G - e$ containing v. This graph is connected, and contains lumps from our set M, so when it is rooted at v, it is also in the set K'. Let be the set of configurations consisting of a rooted point to which is attached some number n of bridges (for $n \geqslant 0$) each of which has a graph from the set K' at the other end. Then the original graph G is obtained by taking a lump from the set M, rooting that lump and attaching to each of its points a configuration from the set $\mathscr{C}$. Here we attach a configuration to a point v by identifying the root point of the configuration with v. The rooted or unrooted status of v is unchanged in the attachment process.

This construction gives a structural characterization of the set K', since any graph obtained in this way is in K'. Thus the graphs in K' are built up recursively from rooted lumps in M, and smaller graphs in K'. In Fig. 3 we see an example of this decomposition in the case that M contains only the graph K_3. The rooted lump $r(G)$ of G is the triangle which contains the root point. The decomposition of G into $r(G)$ and three configurations in $\mathscr{C}$ is shown on the right.

We now translate this structural characterization of the set K' into a functional relation on the appropriate cycle index sums. First, consider the cycle index sum of the set $\mathscr{C}$. Each configuration in $\mathscr{C}$ is either the trivial rooted graph K_1, or else has $n \geqslant 1$ bridges attached to the root. In the former

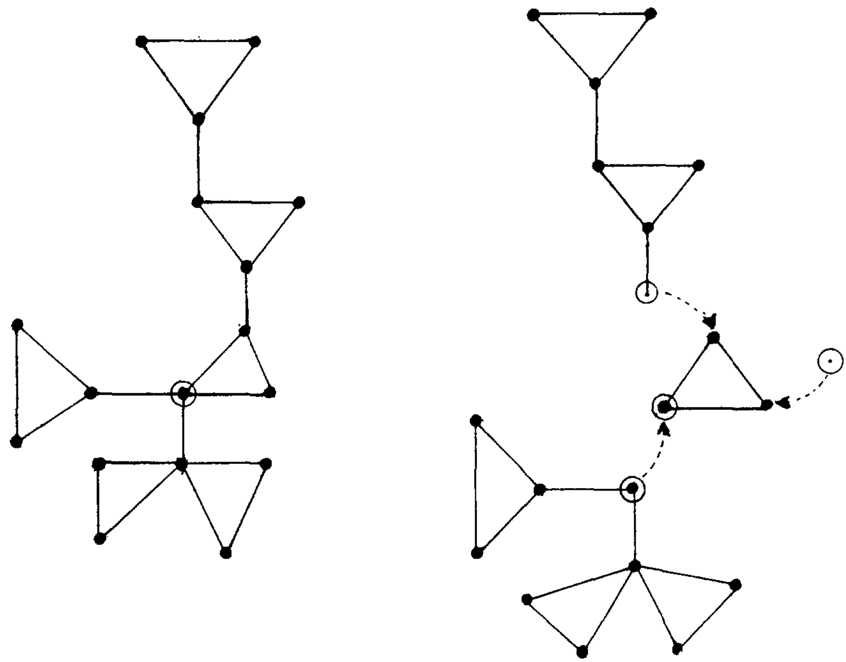

FIG. 3. Obtaining a graph in K' from a graph in M' and configurations in $\mathscr{C}$.

case the cycle index is x_1. In the latter case, since no ordering is imposed on the n bridges the cycle index sum is $x_1(Z(S_n)[Z(K')])$ by the composition theorem. Here the factor of x_1 takes account of the root point, which is always fixed. The sum over n can be accomplished using the identity

$$\sum_{n=0}^{\infty} Z(S_n) = \exp \sum_{i=1}^{\infty} \frac{x_i}{i}, \tag{2.1}$$

which has long been known in the theory of symmetric functions [12, p. 7]. Here $Z(S_0) = 1$, a convention which matches our requirements exactly when $n = 0$. Thus over all $n \geqslant 0$ the sum is

$$Z(C) = x_1 \left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i} [Z(K')]\right). \tag{2.2}$$

Use has been made of the fact that the substitution represented by $[Z(K')]$ is a ring homomorphism on cycle index sums.

Given a rooted lump Λ in M', the composition theorem gives the cycle index sum of all graphs in K' having Λ as the rooted lump as $Z(\Lambda)[Z(\mathscr{C})]$.

Summing over all Λ in M' leads to

$$Z(K') = Z(M')[Z(\mathscr{C})]. \tag{2.3}$$

Using (2.2) gives the desired relation

$$Z(K') = Z(M')\left[x_1\left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i}\,[Z(K')]\right)\right]. \tag{2.4}$$

For example, if M is the one element set containing only the single point graph, then K is the set of all unlabeled trees. Thus $Z(K')$ is the cycle index sum for the set T of rooted trees and Eq. (2.4) becomes

$$Z(T) = x_1 \exp \sum_{i=1}^{\infty} \frac{x_i}{i}\,[Z(T)]. \tag{2.5}$$

The same equation for rooted trees was derived from a different point of view in [17, p. 344]. Replacing x_i by x^i in (2.5) gives the well-known functional equation

$$T(x) = x \exp \sum_{i=1}^{\infty} \frac{1}{i}\,T(x^i).$$

3. Unrooted Connected Graphs with Given Lumps

For a labeled graph on p points there are always p different ways to root it. However, the number of ways to root an unlabeled graph is the number of dissimilar points, which varies in general from 1 to p among the graphs on p points. This is taken into account by a dissimilarity characteristic equation which will enable the cycle index sum for unrooted connected graphs with given lumps to be expressed in terms of the cycle index sum for the given lumps together with the cycle index sum for the rooted connected graphs with the given lumps.

The original dissimilarity characteristic equation for trees was discovered by Otter [13]. It was considerably generalized by Harary and Norman [10], and in this form was used to count graphs with given blocks, as reported by Ford *et al.* [5]. A cycle index sum version of this was obtained by one of the authors [17, Theorem 7] indirectly using calculus. That sort of derivation could be used here, but is avoided in favor of the dissimilarity characteristic approach. The latter is based on the same structural considerations and leads more directly to the desired equation of cycle index sums.

It can be seen that a connected graph must have at least one lump which is adjacent to at most one bridge. Such lumps are called *end lumps* of the graph. If there were no end lumps we could find an endless chain of lumps connected by bridges. In a finite graph this would contain a circuit, which is impossible because its bridge could not disconnect the graph.

The end lumps of a connected graph can be removed in successive layers. It is clear that removal of the end lumps cannot leave a graph with more than one connected component, so at each stage the resulting graph is connected until the empty graph is reached. If at the stage immediately prior to the empty graph we have a single lump, we call this the *central lump.* If at that stage there are two lumps joined by a bridge, we call this bridge the *central bridge.* Every connected graph must contain either a central lump or a central bridge. For a connected graph is nonempty, and therefore contains at least one lump to begin with. If every lump is an end lump then there can not be more than two lumps, and of course if there are two lumps there is just one bridge joining them.

Let G be a connected graph. Two lumps (or two bridges) of G are *similar* if some automorphism of G maps one to the other. If Λ is a lump of G, then G_Λ is the *lump-rooted* graph which results from distinguishing Λ. Thus the automorphisms of G_Λ consist of the automorphisms of G which map Λ to itself. Similarly if b is a bridge of G, then G_b is the *bridge-rooted* graph which results from distinguishing b, and, automorphisms of G_b must fix b. A *symmetry bridge* of G is a bridge the endpoints of which are reversed by some automorphism of G. We can now state our dissimilarity characteristic equation.

THEOREM 1. *If G is a connected graph, then*

$$Z(G)=\sum_{\Lambda} Z(G_\Lambda)-\sum_{b} Z(G_b)+\sum_{s} Z(G_s), \tag{3.1}$$

where Λ ranges over dissimilar lumps of G, b ranges over dissimilar bridges of G (including symmetry bridges, in which case automorphisms of G_b must fix both endpoints of b), and s ranges over symmetry bridges of G (here automorphisms reversing the endpoints of s are included).

Proof. The sum over dissimilar lumps of G is well defined, for $Z(G_\Lambda)=Z(G_{\Lambda'})$ if Λ and Λ' are similar lumps of G, the automorphism groups of G_Λ and $G_{\Lambda'}$ being conjugate subgroups of the full automorphism group of G. For analogous reasons the other two sums are also well defined. We call a lump of G *invariant* if it is similar to no other lump of G. The details of the proof depend on whether G contains an invariant lump.

Case 1. *G contains an invariant lump.*

Let Φ be an invariant lump of G. For a lump Λ of G other than Φ let $b(\Lambda)$ be the bridge adjacent to Λ which a path from Λ to Φ must traverse. Clearly this bridge is unique, for if not, then some pair of bridges adjacent to Λ would be contained in a circuit, contrary to the fact that a bridge must separate G. The function associating $b(\Lambda)$ to Λ is one to one, since Λ is adjacent to two lumps and Λ is not associated with the lump containing interior points of a path from Λ to Φ. Of course Λ is associated with the lump containing the initial point of such a path, so the map is onto. Moreover, because this function is defined structurally with reference to the invariant lump Φ it respects similarity. That is, Λ and Λ' are similar if and only if $b(\Lambda)$ and $b(\Lambda')$ are similar. By the same reasoning G_Λ and $G_{b(\Lambda)}$ are seen to have the same automorphism groups, Λ being mapped to a different lump just if $b(\Lambda)$ is mapped to a different bridge.

It can now be seen that on the right side of (3.1) the sum over Λ other than Φ, and the sum over b add to zero, since they can be rearranged as

$$\sum_{\Lambda \neq \Phi} Z(G_\Lambda) - Z(G_{b(\Lambda)}).$$

The third sum is empty, so the total on the right side is just $Z(G_\Phi)$. However, Φ is invariant under all automorphisms of G, so $Z(G_\Phi) = Z(G)$.

Case 2. *G contains no invariant lump.*

If G contained a central lump, that lump would be invariant. Thus G must instead contain a central bridge s, which is invariant under every automorphism of G. If no automorphism reversed the endpoints of s, then both lumps adjacent to s would be invariant. Thus s must be a symmetry bridge of G. It is easily seen that G can contain no other symmetry bridge, so the third sum gives just $Z(G_s) = Z(G)$. The first two sums total zero much as in Case 1, as each lump is associated with a unique bridge adjacent to it on a path to s, and this association preserves stabilizer subgroups of $\Gamma(G)$, taking into consideration that in the second sum the stabilizer of s consists of automorphism of G which fix each of the endpoints of s. ∎

We now make use of the dissimilarity characteristic theorem to express the cycle index sum $Z(K)$ of all connected graphs with lumps in M in terms of the cycle index sum $Z(M)$ for the lumps in M, and the cycle index sum $Z(K')$ for all rooted connected graphs with lumps in M. This involves summing (3.1) over all G in K, and then rearranging a few of the terms on the right.

THEOREM 3.2.

$$Z(K) = Z(M)\left[x_1\left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i}\,[Z(K')]\right)\right] - \frac{1}{2}Z(K')^2 + \frac{1}{2}x_2[Z(K')]. \quad (3.2)$$

Proof. From the first term on the right of (3.1) we obtain

$\sum_{G \in K} \sum_{\Lambda} Z(G_\Lambda)$, which can be rearranged according to the isomorphism type of Λ to obtain

$$\sum_{\chi \in M} \sum_{G \in K} \sum_{\Lambda \cong \chi} Z(G_\Lambda).$$

For a given χ in M we can view a graph in K which is lump-rooted at a copy of χ as being obtained by attaching connected graphs with lumps in K by bridges to the points of χ. This allows the use of the composition theorem to express the cycle index sum of these graphs as

$$Z(\chi)\left[x_1 \exp \sum_{i=1}^{\infty} \frac{x_i}{i}\,[Z(K')]\right],$$

which is analogous to (2.4). Summing over all χ in M gives the first term on the right of (3.2).

From the second term on the right of (3.1) we obtain $\sum_{G \in K} \sum_b Z(G_b)$. This can be viewed as the sum of cycle indices over configurations obtained by attaching rooted connected graphs with lumps in M to the ends of a specified line where we allow no automorphisms of the resulting graph which interchange the ends of the line. The cycle index sum of such configurations is

$$\tfrac{1}{2} \sum_{\substack{R_1, R_2 \in K' \\ R_1 \neq R_2}} Z(R_1)\, Z(R_2) + \sum_{R \in K'} Z(R)^2. \tag{3.3}$$

Note that (3.3) can be rewritten as

$$\tfrac{1}{2} Z(K')^2 + \tfrac{1}{2} \sum_{R \in K'} Z(R)^2. \tag{3.4}$$

From the third term on the right of (3.1) we obtain $\sum_{G \in K} \sum_S Z(G_S)$. This can be viewed as the sum of the cycle indices of configurations obtained from two copies of the same rooted connected graph R (in K') joined by a bridge between the root points. Such a configuration has

$$\tfrac{1}{2} Z(R)^2 + \tfrac{1}{2} x_2[Z(R)] \tag{3.5}$$

as the cycle index of its automorphism group, which is obtained by composing $Z(S_2)$ over $Z(R)$. Summing (3.5) over all R in K' we obtain

$$\tfrac{1}{2} \sum_{R \in K'} Z(R)^2 + \tfrac{1}{2} x_2[Z(K')]. \tag{3.6}$$

Subtracting (3.4) from (3.6) yields

$$-\tfrac{1}{2} Z(K')^2 + \tfrac{1}{2} x_2[Z(K')].$$

This accounts for the last two terms in (3.2) and proves the theorem. ∎

4. Unlabeled Bridgeless Graphs

In this section the ordinary generating function for the set L of all connected bridgeless graphs will be derived from cycle index sum identities. These identities are obtained from results of Section 3 by using the fact that the set of all connected graphs with lumps in L is just the set C of all connected graphs. Once the ordinary generating function $L(x)$ is determined, it is straightforward to find the numbers of all bridgeless graphs [11, Sect. 4.3]. This is because a graph is bridgeless just if all of its connected components are bridgeless.

From Theorem 3.2 we have

$$Z(C) = Z(L)\left[x_1\left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i}[Z(C')]\right)\right] - \frac{1}{2}Z(C')^2 + \frac{1}{2}x_2[Z(C')]. \quad (4.1)$$

This relation can be inverted as outlined in Section 1. Let $A(x)$ be the unique solution to the equation

$$x = \left(x_1\left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i}[Z(C')]\right)\right)[A(x)] \quad (4.2)$$

having $A(0) = 0$. Composing $A(x)$ into (4.1) and simplifying, we find

$$Z(C)[A(x)] = L(x) - \tfrac{1}{2}(Z(C')[A(x)])^2 + \tfrac{1}{2}x_2[Z(C')[A(x)]]. \quad (4.3)$$

Relation (4.3) has the advantage of giving $L(x)$ in terms of operations on the ordinary generating function $A(x)$ instead of requiring operations on cycle index sums. It is still necessary in principle to substitute into $Z(C)$, but we will show how to determine $Z(C)[A(x)]$ and $Z(C')[A(x)]$ without having direct recourse to $Z(C)$. This will further facilitate the computation of $L(x)$.

Let $D(x) = Z(C')[A(x)]$; then (4.2) takes the form

$$A(x) = x \exp\left(-\sum_{i=1}^{\infty} \frac{1}{i} D(x^i)\right).$$

By separately exponentiating the terms in the coefficients $D_1, D_2, \ldots$ of x, $x^2, \ldots$ in $D(x)$ one finds

$$A(x) = x \prod_{i=1}^{\infty} (1 - x^i)^{D_i}. \quad (4.4)$$

If H is the set of all graphs, then $Z(C)$ is determined from $Z(H)$ by

$$Z(H) = \exp \sum_{i=1}^{\infty} \frac{x_i}{i}[Z(C)]. \quad (4.5)$$

This follows from the composition theorem and identity (2.1); see [11, Eq. (8.4.5); 17, Eq. (10)]. The empty graph is deemed to contribute the constant term 1 to $Z(H)$, but it is not connected and so the constant term in $Z(C)$ is 0. On differentiating one obtains

$$x_1 \frac{\partial}{\partial x_1} Z(H) = Z(H)\, x_1 \frac{\partial}{\partial x_1} Z(C),$$

which by Ford's theorem gives

$$Z(H') = Z(H)\, Z(C'). \tag{4.6}$$

On substituting over $A(x)$ this becomes

$$Z(H')[A(x)] = Z(H)[A(x)]\, D(x). \tag{4.7}$$

The cycle index sum $Z(H)$ for all graphs is easy to compute directly, by a weighted version of Burnside's lemma, as shown in [11, Eq. (7.2.18); 17, Theorem 2]. The necessary relation can be written

$$Z(H) = \sum_{\sigma_1, \sigma_2, \ldots} 2^{\lambda(\sigma_1, \sigma_2, \ldots)} \prod_i (i^{\sigma_i} \sigma_i!)^{-1} x_i^{\sigma_i}$$

with (4.8)

$$\lambda(\sigma_1, \sigma_2, \ldots) = \sum_i \left\{ i \binom{\sigma_i}{2} + \sigma_i \left\lfloor \frac{i}{2} \right\rfloor \right\} + \sum_{i<j} \sigma_i \sigma_j (i, j).$$

Here (i, j) denotes the greatest common divisor of i and j and the first summation is over sequences of nonnegative integers $\sigma_1, \sigma_2, \ldots$ such that only finitely many are nonzero. Again differentiating and applying Ford's theorem, we have

$$Z(H') = \sum_{\sigma_1, \sigma_2, \ldots} \sigma_1 2^{\lambda(\sigma_1, \sigma_2, \ldots)} \prod_i (i^{\sigma_i} \sigma_i!)^{-1} x_i^{\sigma_i}. \tag{4.9}$$

It can now be seen how to determine successively higher coefficients of $D(x)$ and $A(x)$ in a recursive manner. Let $(D(x))_n$ denote $D(x)$ modulo x^n, the terms of $D(x)$ having order less than n, and likewise for any power series. Initially, $(D(x))_1 = 0$. Once $(D(x))_n$ is known, the product in (4.4) can be expanded completely modulo x^n. The leading factor of x then gives $(A(x))_{n+1}$. In turn, (4.8) and (4.9) enable the compositions $(Z(H)[A(x)])_{n+1}$ and $(Z(H')[A(x)])_{n+1}$ to be calculated by using $(A(x))_{n+1}$ in place of $A(x)$ and restricting the sums to sequences satisfying $\sum i\sigma_i \leqslant n$. Finally, these results determine $(D(x))_{n+1}$ by (4.7).

Let $F(x)$ denote $Z(H)[A(x)]$, and note that $(F(x))_{n+1}$ is determined in the course of computing $(D(x))_{n+1}$. Substituting $A(x)$ into both sides of (4.5), we have

$$F(x) = \exp \sum_{i=1}^{\infty} \frac{x_i}{i} [Z(C)[A(x)]].$$

In the usual way one can apply Möbius inversion to the logarithm (Cadogan [4]), obtaining

$$Z(C)[A(x)] = \sum_{i=1}^{\infty} \frac{\mu(i)}{i} \log F(x^i). \tag{4.10}$$

(See [7] for a derivation in a very general setting.) To be explicit, let F_1, F_2,... denote the coefficients of x, x^2,... in $F(x)$, and similarly U_1, U_2,... for $x(\log F(x))' = U(x)$ and V_1, V_2,... for $Z(C)[A(x)] = V(x)$. Since $xF'(x) = F(x)\,U(x)$ and $F(0) = 1$ we find that $F_0 = 1$ and, by comparing the coefficients of x^n on both sides for $n \geqslant 1$,

$$U_n = nF_n - \sum_{i=1}^{n-1} U_i F_{n-i}. \tag{4.11}$$

Then comparison of coefficients in (4.10) gives

$$V_n = \frac{1}{n} \sum_{i \mid n} \mu(i)\, U_i \tag{4.12}$$

for $n \geqslant 1$. (These two equations are a slight variant of [11, Eqs. (4.2.6) and (4.2.9)], with $1 + g(x)$ replaced by $F(x)$ and $c(x)$ by $V(x)$.) Thus from $(F(x))_{n+1}$ we can compute $(V(x))_{n+1}$ by using recurrences (4.11) and (4.12). Finally, in terms of $D(x)$ and $V(x)$, (4.3) can be solved for $L(x)$, resulting in

$$L(x) = V(x) + \tfrac{1}{2}(D(x)^2 - D(x^2)). \tag{4.13}$$

Thus $(L(x))_{n+1}$ is determined at once from $(V(x))_{n+1}$ and $(D(x))_n$.

We illustrate the calculation, starting with $(D(x))_4 = x + x^3 + 4x^4$. By (4.4) we have

$$(A(x))_5 = (x(1-x)(1-x^3)(1-x^4)^4)_5 = x - x^2 - x^4 - 3x^5.$$

From (4.8) the terms of weight $\leqslant 5$ in $Z(H)$ are

$$\begin{aligned} &1 + x_1 + x_1^2 + x_2 + \tfrac{4}{3}x_1^3 + 2x_1x_2 + \tfrac{2}{3}x_3 \\ &\quad + \tfrac{8}{3}x_1^4 + 4x_1^2x_2 + \tfrac{4}{3}x_1x_3 + 2x_2^2 + x_4 \\ &\quad + \tfrac{128}{15}x_1^5 + \tfrac{32}{3}x_1^3x_2 + \tfrac{8}{3}x_1^2x_3 + 8x_1x_2^2 + 2x_1x_4 + \tfrac{4}{3}x_2x_3 + \tfrac{4}{5}x_5. \end{aligned}$$

Then $(Z(H)[A(x)])_5$ is computed by replacing x_1 by $(A(x))_5$, x_2 by $(A(x^2))_5$, etc., and truncating all products to preserve only terms of order $\leqslant 5$. The result is

$$(Z(H)[A(x)])_5 = 1 + x + x^2 + 2x^3 + 4x^4 + 11x^5.$$

Similarly, the terms of weight $\leqslant 5$ in $Z(H')$ are

$$x_1 + 2x_1^2 + 4x_1^3 + 2x_1x_2 + \tfrac{32}{3}x_1^4 + 8x_1^2x_2 + \tfrac{4}{3}x_1x_3$$
$$+ \tfrac{128}{3}x_1^5 + 32x_1^3x_2 + \tfrac{16}{3}x_1^2x_3 + 8x_1^2x_3 + 2x_1x_4,$$

from which one calculates that

$$(Z(H')[A(x)])_5 = x + x^2 + 2x^3 + 7x^4 + 33x^5.$$

Now (4.7) requires

$$\begin{aligned} x + x^2 + 2x^3 + 7x^4 + 33x^5 \\ &= ((1 + x + x^2 + 2x^3 + 4x^4 + 11x^5)(x + x^3 + 4x^4))_5 + D_5x^5, \end{aligned}$$

so $D_5 = 33 - (4 + 1 + 4) = 24$. That is, we now know $(D(x))_5 = x + x^3 + 4x^4 + 24x^5$, which completes one step in the determination of $A(x)$ and $D(x)$.

The subsequent calculation of $(L(x))_5$ is straightforward. We found $(F(x))_5 = 1 + x + x^2 + 2x^3 + 4x^4 + 11x^5$; applying (4.11) one computes $(U(x))_5 = x + \frac{1}{2}x^2 + \frac{4}{3}x^3 + \frac{9}{4}x^4 + \frac{36}{5}x^5$, and then from (4.12) one has $(V(x))_5 = x + x^3 + 2x^4 + 7x^5$. Finally (4.13) is evaluated, yielding

$$\begin{aligned} (L(x))_5 &= x + x^3 + 2x^4 + 7x^5 + \tfrac{1}{2}((x + x^3 + 4x^4)^2 - (x^2))_5 \\ &= x + x^3 + 2x^4 + 7x^5 + x^4 + 4x^5 \\ &= x + x^3 + 3x^4 + 11x^5 + \cdots. \end{aligned}$$

It should be noted that $(Z(H')[A(x)])_{n+1}$ and $(Z(H)[A(x)])_{n+1}$ are very conveniently computed together, since the terms contributing to the former are a subset of the terms contributing to the latter with the rational weight multiplied by σ_1. In practice, of course, one does not store terms in $Z(H)$ but simply runs through the partitions of the numbers up to n, for each partition computing the polynomial contribution and adding it to a running total.

In turns out that the generating function $D(x)$, which was introduced for convenience in describing how to calculate $L(x)$, has a natural combinatorial interpretation. In the case being considered, using all bridgeless connected graphs as lumps so that all connected graphs are obtained, Eq. (2.4) becomes

$$Z(C') = Z(L')\left[x \exp \sum_{i=1}^{\infty} \frac{x_i}{i}\,[Z(C')]\right].$$

Composing both sides over $A(x)$, the defining relation (4.2) for $A(x)$ simplifies the right-hand side, leaving

$$D(x) = Z(L')[x]. \tag{4.14}$$

That is, $D(x)$ is the ordinary generating function for rooted lumps. Together with (4.4) this gives $A(x)$ in an explicit form. It should be noted that the main result of [6] is a combinatorial evaluation of generating functions defined by inversion in a general setting, from which (4.4) and (4.14) follow as a special case.

5. Lines or Bridges as Parameters

In principle, calculating the number $L_{p,q}$ of unlabeled connected bridgeless graphs with p points and q lines is not much harder than calculating L_p, where $L_p = \sum_q L_{p,q}$. We simply work with $\mathbb{Q}[y]$, the ring of rational polynomials in y, in place of $\mathbb{Q}$ as the coefficient domain, and compute the ordinary generating function $L(x, y) = \sum_{p,q} L_{p,q} x^p y^q$ from equations which parallel those used to determine $L(x)$. In practice, of course, this increases the time and space requirements of the computation substantially.

For cycle index sums, we can include lines as a parameter simply by supplying a factor of y^q for each graph with q lines. Thus if S_q denotes the set of graphs in S having exactly q lines. Thus if S_q denotes the set of graphs in S having exactly q lines, then over $\mathbb{Q}[y]$ let

$$Z_e(S) = \sum_{q=0}^{\infty} y^q \sum_{G \in S_q} Z(G).$$

The ordinary generating function $S(x, y)$ by points and lines is given by

$$S(x, y) = Z_e(S)[x]$$

just as before. In general, the composition notation $[Z(S)]$ will denote the replacement of x_i by $Z(S)[y \leftarrow y^i, x_j \leftarrow x_{ij}]$. Then the composition theorem gives the cycle index sum of the Γ-inequivalent functions in Y^X by points and lines as $Z(\Gamma)[Z_e(Y)]$.

The development of the previous sections can now be followed with Z_e in place of Z. The only other change required is to include appropriate powers of y where new lines are being introduced. Using (2.4e) to denote the analogue of (2.4) in which lines are included as a parameter, and so on, the following are obtained:

$$Z_e(K') = Z_e(M')\left[x_1\left(\exp\sum_{i=1}^{\infty}\frac{x_i}{i}[yZ_e(K')]\right)\right], \tag{2.4e}$$

$$Z_e(K) = Z_e(M)\left[x_1\left(\exp\sum_{i=1}^{\infty}\frac{x_i}{i}[yZ_e(K')]\right)\right] - \frac{y}{2}Z_e(K')^2 + \frac{y}{2}x_2[Z_e(K')], \tag{3.2e}$$

$$Z_e(C) = Z_e(L)\left[x_1\left(\exp\sum_{i=1}^{\infty}\frac{x_i}{i}[yZ_e(C')]\right)\right] - \frac{y}{2}Z_e(C')^2 + \frac{y}{2}x_2[Z_e(C')], \tag{4.1e}$$

$$x = \left(x_1\left(\exp\sum_{i=1}^{\infty}\frac{x_i}{i}[yZ_e(C')]\right)\right)[A(x, y)], \tag{4.2e}$$

$$D(x, y) = yZ_e(C')[A(x, y)], \qquad A(x, y) = x\prod_{i=1}^{\infty}\prod_{j=0}^{\binom{i}{2}}(1 - x^i y^j)^{D_{i,j}}, \tag{4.4e}$$

$$yZ_e(H')[A(x, y)] = Z_e(H)[A(x, y)]\,D(x, y), \tag{4.7e}$$

$$P(\sigma_1, \sigma_2, \ldots) = \prod_{i<j}(1 + y^{[i,j]})^{\sigma_i\sigma_j(i,j)}\prod_i(1 + y^i)^{i\binom{\sigma_i}{2} + \lfloor (i-1)/2 \rfloor\sigma_i + \sigma_{2i}},$$

$$Z_e(H) = \sum_{\sigma_1, \sigma_2, \ldots} P(\sigma_1, \sigma_2, \ldots)\prod_i (i^{\sigma_i}\sigma_i!)^{-1} x_i^{\sigma_i}, \tag{4.8e}$$

$$Z_e(H') = \sum_{\sigma, \sigma, \ldots} \sigma_1 P(\sigma_1, \sigma_2, \ldots)\prod_i (i^{\sigma_i}\sigma_i!)^{-1} x_i^{\sigma_i}, \tag{4.9e}$$

$$F(x, y) = Z_e(H)[A(x, y)], \qquad Z_e(C)[A(x, y)] = \sum_{i=1}^{\infty}\frac{\mu(i)}{i}\log F(x^i, y^i), \tag{4.10e}$$

and

$$L(x, y) = \frac{1}{2y}(D(x, y)^2 - D(x^2, y^2)) + \sum_{i=1}^{\infty}\frac{\mu(i)}{i}\log F(x^i, y^i). \tag{4.11e}$$

Here $[i, j]$ denotes the least common multiple of i and j, so that $ij = [i, j](i, j)$. As in Section 4, Eqs. (4.4e), (4.7e), (4.8e), and (4.9e) can be solved for $A(x, y)$, $D(x, y)$, and $F(x, y)$. Then $L(x, y)$ can be calculated from (4.11e). This is the ordinary generating function for unlabeled connected bridgeless graphs, by points and lines. The corresponding generating function for all unlabeled bridgeless graphs is $\exp \sum_{i=1}^{\infty} (1/i) L(x^i, y^i)$.

We now consider the problem of including the number of bridges (instead of the number of lines) as an enumeration parameter. To avoid confusion we take our coefficients in $\mathbb{Q}[z]$ and let the power of z denote the number of bridges. In general, let $Z_b(S)$ denote the cycle index sum of the set S of graphs with the bridges accounted for in powers of z. The same composition convention applies to $Z_b(S)$ as for $Z_e(S)$ with y replaced by z throughout. Equations (2.4e), (3.2e), and (4.1e) can at once be converted to Eqs. (2.4b), (3.2b), and (4.1b) by replacing Z_e by Z_b and y by z. This is because all of the new lines in the former equations are bridges. However, the equations are now to be solved in the reverse direction, since $Z_b(H)$ is not known and $Z_b(L) = Z(L)$ can be found as in the previous section. In the context of all bridgeless connected graphs as lumps, Eq. (2.4b) takes the form

$$Z_b(C') = Z(L') \left[x_1 \left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i} [Z_b(C')] \right) \right]. \tag{5.1}$$

In principle, given $Z(L)$ we can at once apply $x_1(\partial/\partial x_1)$ to compute $Z(L')$, and then find $Z_b(C')$ and $x_1(\exp \sum_{i=1}^{\infty} (x_i/i)[zZ_b(C')])$ recursively using (5.1). These along with $Z(L)$ could be combined according to (4.1b) to give $Z_b(C)$.

The last step can be simplified considerably if only the ordinary generating function $C(x, z)$ of unlabeled connected graphs by points and bridges is desired. Let

$$D(x, z) = zZ_b(C')[x]$$

and

$$V(x, z) = x \exp \sum_{i=1}^{\infty} \frac{1}{i} D(x^i, z^i).$$

Then composing both sides of (4.1b) over $[x]$ gives

$$C(x, z) = Z(L)[V(x, z)] - \frac{1}{2z} (D(x, z)^2 - D(x^2, z^2)). \tag{5.2}$$

Of course the ordinary generating function $H(x, z)$ for all unlabeled graphs by numbers of points and bridges is given by

$$H(x, z) = \exp \sum_{i=1}^{\infty} \frac{1}{i} C(x^i, z^i), \tag{5.3}$$

as is immediate from (4.5b). The solution by recurrence relations obtained in (4.11) and (4.12) for solving a similar equation can be imitated here with z as an additional variable. Setting $U(x, z) = x \sum_{i=1}^{\infty} (\partial/\partial x)\, C(x^i, z^i)$, one has

$$U_{p,q} = \sum_{i \mid (p,q)} \frac{p}{i} C_{p/i,q/i}. \tag{5.4}$$

On applying $x(\partial/\partial x)$ to both sides of (5.3) and equating coefficients of $x^p y^p$, we see that

$$H_{p,q} = \frac{1}{p} \sum_{i=0}^{p-1} \sum_{j=0}^{i-1} H_{i,j} U_{p-i,q-j} \tag{5.5}$$

for $p \geqslant 1$. With the initial condition $H(0, z) = 1$, (5.4) and (5.5) determine $H(x, z)$ from $C(x, z)$ very efficiently.

The calculation of $C(x, z)$ can be simplified still further by applying the method of Section 4 to the computation of $Z(L)[V(x, z)]$. To start, let $A(x, z)$ be the unique solution of

$$V(x, z) = \left(x_i \left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i} [Z(C')]\right)\right) [A(x, z)] \tag{5.6}$$

such that $A(0, z) = 0$. Setting

$$E(x, z) = Z(C')[A(x, z)],$$

(5.6) can be rewritten in the form

$$A(x, z) = V(x, z) \prod_{i=1}^{\infty} \prod_{j=0}^{i-1} (1 - x^i z^j)^{E_{i,j}}. \tag{5.7}$$

The other relation between $A(x, z)$ and $E(x, z)$ is

$$Z(H')[A(x, z)] = Z(H)[A(x, z)]\, E(x, z). \tag{5.8}$$

In computing $A(x, z)$ and $E(x, z)$ recursively from (5.7) and (5.8) we obtain $Z(H)[A(x, z)]$ as a by-product. Finally, we have

$$Z(L)[V(x, z)] = \frac{1}{2} E(x, z)^2 - \frac{1}{2} E(x^2, z^2) + \sum_{i=1}^{\infty} \frac{\mu(i)}{i} \log Z(H)[A(x, z)]. \tag{5.9}$$

Equations (5.6)–(5.9) are analogous to (4.2), (4.4), (4.7), and (4.13), respectively, and are derived and solved in exactly the same way.

To complete the calculation of $C(x, z)$ and $H(x, z)$ without recourse to explicit manipulation of cycle index sums, it remains to determine $V(x, z)$ in some such manner. Composing (5.1) over $[x]$ and multiplying by z, the definitions of $V(x, z)$ and $D(x, z)$ along with the associativity of composition give

$$D(x, z) = zZ(L')[V(x, z)]. \tag{5.10}$$

On the other hand, as a special case of (2.4) we have

$$Z(C') = Z(L')\left[x_1\left(\exp \sum_{i=1}^{\infty} \frac{x_i}{i}\,[Z(C')]\right)\right].$$

Composing both sides of this equation over $A(x, z)$ and applying (5.6) yields

$$Z(C')[A(x, z)] = Z(L')[V(x, z)].$$

By (5.10) and the definition of $E(x, z)$, then,

$$D(x, z) = zE(x, z). \tag{5.11}$$

This allows us to express the definition of $V(x, z)$ in terms of $E(x, z)$, and then combine with (5.7) to give $A(x, z)$ directly from $E(x, z)$. In the product formulation the relation is

$$A(x, z) = x \prod_{i=1}^{\infty} \prod_{j=0}^{i} (1 - x^i x^j)^{E_{i,j} - E_{i,j-1}}. \tag{5.12}$$

Now (5.8) and (5.12) can together be recursively solved for $E(x, z)$ and $A(x, z)$. In the process $Z(H)[A(x, z)]$ will also be determined. Denoting this generating function by $F(x, z)$, we then have

$$Z(C)[A(x, z)] = \sum_{i=1}^{\infty} \frac{\mu(i)}{i} \log F(x^i, z^i) \tag{5.13}$$

by analogy with (4.10). The latter can be combined with Eq. (5.2), (5.9), and (5.11) to give

$$C(x, z) = \frac{1-z}{2}\,(E(x, z)^2 - E(x^2, z^2)) + \sum_{i=1}^{\infty} \frac{\mu(i)}{i} \log F(x^i, z^i). \tag{5.14}$$

This relation finally allows the calculation of $C(x, z)$ without manipulation cycle index sums.

If desired, one can combine the methods of this section to give equations by which unlabeled graphs can be enumerated by points, lines and bridges. The details are omitted because they are entirely straightforward.

6. Labeled Bridgeless Graphs

Relations among the exponential generating functions for labeled graphs can be deduced directly from relations among the cycle index sums for unlabeled graphs. The observations needed for this deduction are presented as three propositions. In these propositions let S, T, and U be sets of unlabeled graphs, let s, t, and u denote the sets of labeled graphs whose unlabeled versions lie in S, T, and U, and let $s(x)$, $t(x)$, and $u(x)$ be the exponential generating functions for s, t, and u.

Proposition 6.1.

$$s(x) = Z(S)[x_1 \leftarrow x, x_{i+1} \leftarrow 0].$$

Proof. Let G be a graph in S, with g labeled versions and p points. The contribution of the labeled versions of G to $s(x)$ is just $gx^p/p!$. The replacement of x_{i+1} by 0 for all $i > 0$ in $Z(\Gamma(G))$ leaves only the term $x^p/|\Gamma(G)|$ contributed by the identity of $\Gamma(G)$, and the replacement of x_1 by x leaves $x^p/|\Gamma(G)|$ as the contribution of G to $Z(S)[x_1 \leftarrow x, x_{i+1} \leftarrow 0]$. Now $g = p!/|\Gamma(G)|$ since there is obviously a 1–1 correspondence between labeled versions of G and left cosets of $\Gamma(G)$ in the symmetric group S_p. Thus the contributions of G to the two sides of the equation are equal. The result follows by summing over all G in S. ∎

Proposition 6.2. *If* $Z(S) = Z(T)[Z(U)]$, *then* $s(x) = t(u(x))$.

Proof. Observe that the subscript of every variable in $x_{i+1}[Z(U)]$ is a multiple of $i+1$. Thus

$$\begin{aligned}(Z(T)[Z(U)])&[x_1 \leftarrow x, x_{i+1} \leftarrow 0]\\ &= Z(T)[x_1 \leftarrow Z(U)[x_1 \leftarrow x, x_{i+1} \leftarrow 0], x_{i+1} \leftarrow 0],\end{aligned}$$

and the result follows by Proposition 6.1. ∎

Proposition 6.3. *If* $Z(S) = x_1(\partial/\partial x_1)\, Z(T)$, *then* $s(x) = x(d/dx)\, t(x)$.

Proof. By the chain rule

$$Z(S)[x_1 \leftarrow x, x_{i+1} \leftarrow 0] = x\frac{d}{dx} Z(T)[x_1 \leftarrow x, x_{i+1} \leftarrow 0],$$

so again the result follows by Proposition 6.1. ∎

Let $c(x)$, $c'(x)$, $l(x)$, and $l'(x)$ be the exponential generating functions for labeled connected graphs which are unrestricted, rooted, bridgeless, and rooted and bridgeless, respectively. Either directly, or else combining Proposition 6.3 with Ford's theorem, we have $c'(x) = x(d/dx)\,c(x)$ and $l'(x) = x(d/dx)\,l(x)$.

With $l(0) = 0$ it is easy to find $l(x)$ from $l'(x)$. Relation (2.4) in the case of all connected graphs and all connected bridgeless graphs can be reduced by using Propositions 6.1–6.3 to give

$$c'(x) = l'(x \exp c'(x)). \tag{6.1}$$

Now $c(x)$ is easily found from the relation

$$c(x) = \log \sum_{i=0}^{\infty} (2^{\binom{i}{2}} x^i/i!),$$

which results from (4.5) and Proposition 6.1. Then (6.1) can be solved for $l'(x)$ by series reversion. Alternatively, we can obtain $l(x)$ by reversion of

$$l(x \exp c'(x)) = c(x) + \tfrac{1}{2} c'(x)^2, \tag{6.2}$$

which is obtained from (4.1) in a similar manner.

If lines or bridges are to be included as enumeration parameters, the same procedures can be applied to the results of the previous section. Here it is only the points which are treated in the exponential fashion, so that, for instance, the coefficient of $x^p y^q/p!$ in $l(x, y)$ is the number of labeled connected bridgeless graphs with exactly p points and q lines. Then $c(x, y)$ is determined from

$$c(x, y) = \log \sum_{i=0}^{\infty} ((1 + y)^{\binom{i}{2}} x^i/i!),$$

which follows from (4.5e) and Proposition 6.1. In turn, $l'(x, y)$ can be obtained by reversion from

$$c'(x, y) = l'(x \exp(yc'(x, y)), y). \tag{6.3}$$

Again there is an alternative based on (4.1e), to determine $l(x, y)$ by solving

$$l(x \exp(yc'(x, y)), y) = c(x, y) + \frac{y}{2} c'(x, y)^2. \tag{6.4}$$

If bridges are to be included, then one first finds $l'(x)$. From (5.1) and the Propositions 6.1–6.3 one has

$$c'(x, z) = l'(x \exp(zc'(x, z))). \tag{6.5}$$

The latter is then solved for $c'(x, z)$, and hence $c(x, z)$. Of course the exponential generating function $h(x, z)$ of all labeled graphs by points and bridges is just given by

$$h(x, z) = \exp c(x, z).$$

Finally, one can employ (5.2) in place of (5.1) to find the relation

$$l(x \exp(zxc_x(x, z))) = c(x, z) + \frac{z}{2} x^2 (c_x(x, z))^2. \tag{6.6}$$

Here $c_x(x, z)$ is the partial derivative of $c(x, z)$ with respect to x. Knowing $l(x)$, Eq. (6.6) can be solved for $c(x, z)$ and $c_x(x, z)$.

An improved equation, implicit in [19], is obtained by combining the two. Differentiating both sides of (6.6) with respect to z gives

$$\begin{aligned} c_z(x, z) = {} & l_x(x \exp(zxc_x(x, z)))(x \exp(xzc_x(x, z)))(xc_x(x, z) + zxc_{xz}(x, z)) \\ & - \frac{x^2}{2} (c_x(x, z))^2 - zx^2 c_x(x, z)\, c_{xz}(x, z). \end{aligned} \tag{6.7}$$

Using Eq. (6.5) we can rewrite this as

$$\begin{aligned} c_z(x, z) = {} & xc_x(x, z)(xc_x(x, z) + zxc_{xz}(x, z)) - \frac{x^2}{2} (c_x(x, z))^2 \\ & - zx^2 c_x(x, z)\, c_{xz}(x, z). \end{aligned} \tag{6.8}$$

The latter simplifies considerably to the form

$$c_z(x, z) = \tfrac{1}{2} x^2 c_x(x, z). \tag{6.9}$$

Along with the boundary condition $c(x, 1) = c(x)$, the partial differential equation (6.9) determines $c(x, z)$ in terms of $c(x)$. The corresponding pair of recurrence relations for the coefficients of $c(x, z)$ is presented in [19] along with an indication of a very direct combinatorial interpretation.

There is also a more direct method to compute the number of labeled bridgeless graphs by points and lines, based on a differential equation satisfied by $l(x, y)$. To start, we need to derive a differential equation for $c(x, y)$.

Let $h(x, y)$ denote the exponential generating function for all graphs by numbers of points and lines, so that

$$h(x, y) = \sum_{n=0}^{\infty} ((1 + y)^{\binom{n}{2}} x^n / n!).$$

Differentiating, we find

$$h_x(x, y) = h(x(1+y), y)$$

and

$$h_y(x, y) = \tfrac{1}{2}x^2 h(x(1+y)^2, y).$$

As noted earlier $c(x, y) = \log h(x, y)$, so one has

$$c_x(x, y) = \frac{h(x(1+y), y)}{h(x, y)} \tag{6.10}$$

and

$$c_y(x, y) = \frac{x^2 h(x(1+y)^2, y)}{2h(x, y)}. \tag{6.11}$$

Differentiating (6.10) with respect to x and simplifying by use of (6.10) and (6.11), we find

$$\frac{x^2}{2}(c_{xx}(x, y) + c_x(x, y)^2) = (1+y)\, c_y(x, y). \tag{6.12}$$

This equation appeared in [23], and the equivalent recurrence relation in [26, Eq. (3)]. In both cases the proof given was combinatorial.

Letting $u = x\exp(xyc_x(x, y))$, we can write (6.3) as

$$xc_x = ul_x(u). \tag{6.13}$$

Here $l_x(u)$ stands for $l_x(u, y)$ and c_x for $c_x(x, y)$. In the same notation, (6.4) yields

$$l(u) = c + \frac{yx^2}{2} c_x^2.$$

Differentiating with respect to y and using (6.13) to simplify, one finds

$$c_y = l_y(u) + \tfrac{1}{2}u^2 l_x(u)^2. \tag{6.14}$$

Finally, differentiating (6.13) with respect to x and simplifying as before gives

$$\begin{aligned} x^2 c_{xx}(1 - yul_x(u) - yu^2 l_{xx}(u)) \\ = u^2 l_{xx}(u) + yu^2 l_x(u)^2 + yu^3 l_x(u)\, l_{xx}(u). \end{aligned} \tag{6.15}$$

If we now multiply (6.12) through by $(1 - yul_x(u) - yu^2 l_{xx}(u))$, then

(6.13)–(6.15) can be used to express the result entirely in terms of y, u, $l_y(u)$, $l_x(u)$, and $l_{xx}(u)$. It is then valid to replace u by x throughout, leading to

$$\begin{aligned} l_y &= \tfrac{1}{2} y^2 x^3 l_x^3 + \tfrac{1}{2} x^2 l_{xx} - y l_y + xy(1+y)\, l_y (l_x + x l_{xx}) \\ &\quad + \tfrac{1}{2} y x^3 l_{xx} l_x (1 + y x l_x). \end{aligned} \tag{6.16}$$

Comparing coefficients of $x^p y^q / p!$ on both sides, one finds $(q+1)\, l_{p,q+1}$ on the left in terms of numbers $l_{i,j}$ on the right, where $i \leqslant p$ and $j \leqslant q$. Given the obvious initial conditions $l_{1,0} = 1$ and $l_{i,0} = 0$ for $i > 1$, the resulting recurrence can be used to solve for any $l_{p,q}$.

It is not hard to see that when points and bridges are both required as enumeration parameters it is more efficient computationally to use (6.9) than either (6.5) or (6.6) to find $c(x, z)$. Likewise, if lines are required as a parameter, then (6.16) is a better basis for computation than (6.3) or (6.4). Apparently the improvements offered by these differential equations are not available if only points are needed as a parameter. Moreover the method seems to have no application in the enumeration of unlabeled graphs.

7. Numerical Results

This section contains numerical results obtained by the authors from the equations in Section 4. The programming was done by Hanlon on an IBM370. The authors are indebted to the department of mathematics at Caltech for supporting the computing costs.

TABLE I

$A(x)$ mod 12007

n	a_n (mod 12007)
0	0
1	1
2	−1
3	0
4	−1
5	−3
6	−20
7	−169
8	−2223
9	−9445
10	−3450
11	−92
12	−8268

TABLE II

Rooted Unlabeled Bridgeless Graphs

d_n	n
1	1
0	2
1	3
4	4
24	5
193	6
2420	7
47912	8
1600524	9
93253226	10
9694177479	11
1822463625183	12
625829508087155	13
395785845695978077	14
464137111800208818956	15
1015091598240432264958267	16
4160447480034069826186309689	17
32088552194861245127627790541334	18
467409601117828706798569588745772153	19
12899018653180446597120165915370010416191	20
676315270437729957020074803660041084140580169	21
67541678379430249600075434275751575939772386952165	22

The general algorithm was that outlined in Section 4. All computations were done modulo a prime and the actual numbers were recovered at the end using the Chinese remainder theorem. The modular arithmetic facilitated the programming as it avoided the necessity to build special packages to handle large integer computations.

The series $A(x)$ and $F(x)$ were not needed explicitly, hence were only computed modulo primes. The series $A(x)=\sum_{n=1}^{\infty} a_n x^n$ modulo 12007 is given in Table I. The final numerical values for the coefficients of $L(x)=\sum_{n=1}^{\infty} l_n x^n$ and $D(x)=\sum_{n=1}^{\infty} d_n x^n$ appear in Tables II and III (recall that l_n is the number of unlabeled bridgeless graphs with n points, and d_n is the number of unlabeled, rooted bridgeless graphs with n points).

8. Related Results and Problems

For labeled nonseparable graphs by numbers of points and lines there are differential equations along the lines of (6.16). These are derived in [23, 25]

TABLE III

Unlabeled Bridgeless Graphs

l_n	n
1	1
0	2
1	3
3	4
11	5
60	6
502	7
7403	8
197442	9
9804368	10
902818087	11
153721215608	12
48443044675155	13
28363687700395422	14
30996524108446916915	15
63502033750022111383196	16
244852545022627009655180986	17
1783161611023802810566806448531	18
24603891215865809635944516464394339	19
644997736409807527763636776555094120938	20
32206723222694820999428520409229664177815561	21
3070137798431519340432448500050636943651710712237	22

on the basis of a structuaral characterization of the graphs obtained by deleting a line from a nonseparable graph. For labeled 3-connected graphs by numbers of points and lines a similar analysis has been carried out in [23, 24], though in this case it is much more difficult. Enumerations of 3-connected graphs, and of nonseparable graphs with no points of degree two, have also been obtained in a more traditional manner which does not require the number of lines as a parameter, in [20, 21].

The cycle index sum version techniques of this paper were developed originally to aid in counting unlabeled graphs without points of degree one or two, and unlabeled nonseparable graphs. Details and extensive numerical results will be available in [18]. These methods have also been applied by the authors to the enumeration of unlabeled graphs which are 3-connected and 3-line-connected [8].

It is well known that for fixed k, most labeled graphs asymptotically are k-connected; see [2, Chap. 7; 19]. The same is true for unlabeled graphs with p points as $p \to \infty$, as shown in [22]. The latter paper also shows the surprising fact that for unlabeled graphs with no isolates on q lines, most are connected but not 2-connected as $q \to \infty$. The same proof shows that most such graphs contain at least one bridge, and in fact at least one endpoint, as $q \to \infty$.

An unsolved problem is to count k-connected or k-line-connected graphs for any $k \geqslant 4$. This appears to be very difficult, either for labeled or unlabeled graphs.

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