The Asymptotic Number of Claw-free Cubic Graphs*

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Abstract

Let H_n be the number of claw-free cubic graphs on $2n$ labeled nodes. In an earlier paper we characterized claw-free

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cubic graphs and derived a recurrence relation for H_n. Here we determine the asymptotic behavior of this sequence:

$$H_n \sim \frac{(2n)!}{e\sqrt{6\pi n}} \left(\frac{n}{2e}\right)^{n/3} e^{(n/2)^{1/3}}.$$

We have verified this formula using known asymptotic estimates of cubic graphs with loops and multiple edges and also by the method of inclusion and exclusion.

1 Introduction

A graph is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. Clearly a cubic graph is claw-free if and only if every vertex is contained in a triangle. At the 1992 Kalamazoo conference at Western Michigan University, M. D. Plummer (see [Pl95]) raised a number of questions about the enumeration of claw-free cubic graphs and asked for the probabilistic behavior of hamiltonicity in claw-free cubic graphs, in the planar case and in general. The latter problem was solved (based on results in the present paper) in [RoW01] where it was shown that almost all claw-free cubic graphs are hamiltonian. In our first paper [PaRR0x] approaching these problems, we used combinatorial reductions to derive a second-order, linear homogeneous equation with polynomial coefficients whose power series solution is the exponential generating function for claw-free cubic graphs. From this we derived the following recurrence relation for H_n, the number of labeled claw-free cubic graphs of order $2n$:

$$\begin{aligned} H_{n+1} &= (6n-5)\binom{2n+1}{3} H_{n-1} + 60(2n^2-7)\binom{2n+1}{5} H_{n-2} \\ &+ 420(12n-31)\binom{2n+1}{7} H_{n-3} - 60480(4n-19)\binom{2n+1}{9} H_{n-4} \\ &- 3326400(6n^2-54n+127)\binom{2n+1}{11} H_{n-5} \\ &- 172972800(9n^2-108n+347)\binom{2n+1}{13} H_{n-6} \end{aligned}$$

$$- 54486432000(n-1)\binom{2n+1}{15} H_{n-7}$$
$$+ 59281238016000(n-7)\binom{2n+1}{17} H_{n-8}$$
$$+ 422378820864000(18n-97)\binom{2n+1}{19} H_{n-9}$$
$$+ 6563766876226560000\binom{2n+1}{21} H_{n-10}$$
$$+ 67322960257512960000\binom{2n+1}{23} H_{n-11}. \qquad (1.1)$$

Of course $H_{n-j} = 0$ whenever $j > n$. With the initial conditions $H_0 = 1$ and $H_1 = 0$, (1.1) can be used to compute the values of $H_2, \cdots, H_{n+1}$ using just $O(n)$ arithmetic operations each. In this way we computed the values shown in Table 2 of [PaRR0x], where one finds, for example, that H_{26} is

$$1016031492424337300070147499566814430489390287664828\\ 5295864422890087890625. \qquad (1.2)$$

In this paper, which forms the next stage of our approach to Plummer's problems, we determine the asymptotic behavior of H_n.

Theorem 1.1. The number H_n of labeled claw-free cubic graphs of order $2n$ has the asymptotic value:

$$H_n \sim \frac{(2n)!}{e\sqrt{6\pi n}} \left(\frac{n}{2e}\right)^{n/3} e^{(n/2)^{1/3}}. \qquad (1.3)$$

We will derive the asymptotic result of the theorem in two ways. The first depends on the characterization of claw-free cubic graphs developed in [PaRR0x] as well as the known asymptotic behavior of cubic general graphs (see [BeC78],[W79a] or [CPR0x]). The second method takes a more direct approach using inclusion and exclusion, which eliminates considerable fussing over negligible contributions.

For general graph-theoretic terminology we use [HP73] except for adopting the more conventional names "nodes and edges" instead of "points and lines". We assume a basic knowledge of labeled enumeration techniques using *egf*'s, such as is found in Chapter 1 of [HP73], as well as the terminology developed in [PaRR0x] used to characterize claw-free cubic graphs. From here on we frequently refer to the later as *cfc*'s. In a *cfc* a node may be in exactly two triangles precisely if it is a node of degree 3 in an induced subgraph isomorphic to $K_4 - e$; we call such a subgraph a *diamond.* A maximal set of diamonds which are adjacent in series is called a string of diamonds. A connected graph of order at least 5 in which every node is contained in a diamond is called a *ring of diamonds.*

2 Formulas for cubics and claw-free cubics

The notions of *dilation* and *expansion* of cubic general graphs, explained fully in [PaRR0x], form the basis of our characterization of *cfc*'s. Roughly speaking, a general cubic G is *dilated* by replacing each node u by a triangle, i.e. three new mutually adjacent nodes, say u_a, u_b and u_c. If u and v are adjacent in G, then one of $\{u_a, u_b, u_c\}$ is adjacent to one of $\{v_a, v_b, v_c\}$ in the dilation G'. As a consequence, G' is a cubic multigraph in which each node belongs to a triangle and the only multiple edges come from loops in G. Each loop in G gives rise to an instance of the configuration known as a *trumpet* in G'. Next G' is *expanded* by placing a string of at least one diamond on just one edge of every double edge of all the trumpets and arbitrary (possibly empty) strings of diamonds on the edges of G' that correspond to original edges of G. The end result is a graph G'' which is a *cfc*. Let G_n be the number of labeled *cfc*'s of order $2n$ that can be built in this way, i.e. by dilating and expanding general cubic graphs which have no components isomorphic to the triple edge of order 2. Our next goal is to determine the asymptotic behavior of G_n.

Let $g(2m, \ell, d)$ be the number of labeled cubic general graphs of order $2m$ with ℓ loops, d double edges and no triple edges. Then if s is the number of single edges in such a graph, of course

$$3(2m) = 2s + 4d + 2\ell.$$

Then we define $F(2n, \ell, d)$ to be the number of *cfc*'s of order $2n$ built from general cubics with ℓ loops, d double edges and no triple edges by dilating vertices and expanding edges. For fixed n, ℓ and d, we have

$$F(2n,\ell,d) \;=\; \sum g(2m,\ell,d)\binom{2n}{6m}\frac{\binom{6m}{3,\ldots,3}}{(2m)!}(3!)^{2m-\ell-2d}(3^2\cdot 2)^d 3^\ell$$
$$\left(\!\!\binom{3m}{j-\ell}\!\!\right)\binom{4j}{4,\ldots,4}12^j, \tag{2.1}$$

where the sum is over all values of m and j with $2n = 3(2m)+4j$ and $j \geq \ell$. Here is a sketch of the justification of this formula. Suppose G is a cubic general graph counted by $g(2m,\ell,d)$. First we choose $6m$ labels from the $2n$ available and we arrange them in $2m$ unordered groups of three each for dilation. The number of ways to do this is

$$\binom{2n}{6m}\binom{6m}{3,\ldots,3}/(2m)!.$$

Then it can be seen that the number of ways to form the adjacencies in a dilation is

$$(3!)^{2m-\ell-2d}((3!)^2/2)^d\cdot 3^\ell$$

following the pattern established by G. Since a diamond must be assigned to each loop, there are $j-\ell \geq 0$ remaining which can be strung on any of the $3m$ original edges of the dilation in $\left(\!\binom{3m}{j-\ell}\!\right)$ ways. Note that we use the repeated parentheses to indicate combinations with repetition. Now there are $\binom{4j}{4,\ldots,4}$ ways to arrange the remaining labels in groups for the diamonds and 12 ways to assign labels to each of the j groups. On applying the multiplication principle, we arrive at (2.1).

Now G_n can be expressed in terms of the $F(2n,\ell,d)$:

$$G_n = \sum F(2n,\ell,d), \tag{2.2}$$

where the sum is over all relevant values of ℓ and d.

To evaluate G_n asymptotically we begin with a lemma for $F(2n,\ell,d)$.

Lemma 2.1. For both $\ell, d = o(\sqrt{n})$,

$$F(2n,\ell,d) \sim \frac{e^{-2}}{\ell!d!}\left(\left(\frac{2}{n}\right)^{2/3}/2\right)^{\ell}\frac{(2n)!}{\sqrt{6\pi n}}\left(\frac{n}{2e}\right)^{n/3}e^{(n/2)^{1/3}}.$$

Proof. The basis for this result is the following formula for $g(2m,\ell,d)$ which holds for both ℓ and $d = o(\sqrt{m})$:

$$g(2m,\ell,d) \sim \frac{e^{-2}}{\ell! d!} \, \frac{(6m)!}{2^{3m}(3m)!} \div \left\{(3!)^{2m-\ell-2d}(3^2\cdot 2)^d \cdot 3^\ell\right\}. \qquad (2.3)$$

It can be derived using inclusion and exclusion on two types of properties for loops and double edges (see [CPR0x]).

It can also be shown that for all ℓ and d

$$g(2m,\ell,d) = O(1)\frac{(6m)!}{2^{3m}(3m)!}\,\frac{1}{(3!)^{2m}}\,\frac{2^\ell 2^d}{\ell! d!}$$

and hence the total number of general cubic graphs with $2m$ nodes is dominated by the graphs for which (2.3) holds. It requires some more work to show that (2.3) holds for the summands which dominate in (2.1). Then on substituting the right side of (2.3) in (2.1) and eliminating m we obtain

$$F(2n,\ell,d) \sim \frac{e^{-2}}{\ell! d!}\sqrt{\frac{3}{2\pi}}(2n)!\sum\left(\frac{n-2j}{2e}\right)^{\frac{n-2j}{3}} \frac{1}{(n-2j)^{1/2}}\binom{n-j-\ell-1}{j-\ell}\left(\frac{1}{2}\right)^j, \qquad (2.4)$$

where the sum is over all $j \ge \ell$ with $2n = 3(2m)+4j$, as above in (2.1).

An application of the ratio test on the right side of (2.4) shows that the sum peaks for j near $(n/2)^{1/3}$. And for $j = O((n/2)^{1/3})$ we have

$$\left(\frac{n-2j}{2e}\right)^{\frac{n-2j}{3}}\frac{1}{(n-2j)^{1/2}} \sim \frac{1}{\sqrt{n}}\left(\frac{n}{2e}\right)^{n/3}\left(\frac{2}{n}\right)^{2j/3}$$

and

$$\binom{n-j-\ell-1}{j-\ell} \sim \frac{n^{j-\ell}}{(j-\ell)!}.$$

Some calculation shows that the sum in (2.4) is dominated by the terms for which $j \le 3n^{1/3}$. And the extra terms added in (2.5) below are seen to be negligible.

This implies

$$F(2n,\ell,d) \sim \frac{e^{-2}}{\ell!d!}\left(\left(\frac{2}{n}\right)^{2/3}/2\right)^{\ell}\sqrt{\frac{3}{2\pi n}}(2n)!\left(\frac{n}{2e}\right)^{n/3}$$
$$\sum_{j\geq\ell}\frac{\left((n/2)^{1/3}\right)^{j-\ell}}{(j-\ell)!}. \qquad (2.5)$$

Notice that the sum is over those values of j that satisfy the equation $2n = 3(2m) + 4j$ and hence $n - 2j \equiv 0 \pmod 3$. For example if n is a multiple of 3 and $\ell = 0$, then the sum in (2.5) retains every third term in the exponential series for $(n/2)^{1/3}$. The extraction from a power series of every r-th term is fully explained in Wilf's book [Wi90] (see pp. 47-48) and all of the details are given for the case we need with $r = 3$. As a consequence, the sum in (2.5) is asymptotic to $(\exp((n/2)^{1/3}))/3$ from which the formula in the lemma follows. □

On applying the lemma to formula (2.2) for G_n together with justification for excluding negligible terms, we have

$$G_n \sim e^{-2}\frac{(2n)!}{\sqrt{6\pi n}}\left(\frac{n}{2e}\right)^{n/3} e^{(n/2)^{1/3}}\sum_{\ell,d}\frac{\left(\left(\frac{2}{n}\right)^{2/3}/2\right)^{\ell}}{\ell!d!}. \qquad (2.6)$$

But the sum in (2.6) is just

$$\exp\left\{1+\left(\frac{2}{n}\right)^{2/3}/2\right\},$$

and so

$$G_n \sim e^{-1}\frac{(2n)!}{\sqrt{6\pi n}}\left(\frac{n}{2e}\right)^{n/3} e^{(n/2)^{1/3}}.$$

Note that the contribution to these *cfc*'s built from cubic general graphs with loops is negligible.

Let B_n denote the number of claw-free cubic graphs of order $2n$ whose components consist of K_4's, rings of diamonds and dilations and expansions of triple edges. We call these *exotic* components and set $B_0 = 1$. Of course $B_1 = 0$, $B_2 = 1$ and $B_3 = 60$. Now the total number of *cfc*'s can be expressed in terms of the B_n and G_n:

$$H_n = \sum_{k=0}^{n} \binom{2n}{2k} B_k G_{n-k}. \tag{2.7}$$

Our next task is to show that the number of *cfc*'s with exotic components is negligible, i.e.

$$H_n \sim G_n. \tag{2.8}$$

Let $\varphi(z)$ be the *egf* for *cfc*'s whose components are all exotic, i.e.

$$\varphi(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{(2n)!}.$$

In [PaRR0x] we showed that the contributions to $\varphi(z)$ made by the components isomorphic to K_4's and rings of diamonds is

$$\sqrt{b}\exp(-5z^2/24),$$

where

$$b = b(z) = (1 - z^2/2)^{-1}.$$

In the same paper it was shown that the *egf* for components derived from triple edges is

$$\exp(z^3 b^3/12).$$

Thus

$$\varphi(z) = \sqrt{b}\exp(-5z^2/24 + (zb)^3/12).$$

Since $\varphi(z)$ is regular in the complex open disk $|z| < \sqrt{2}$, we know

$$B_k/(2k)! = o(c^k)$$

for any $c > 1/\sqrt{2}$. Hence $B_k/(2k)!$ is bounded above for all k by c^k times a suitable constant. Similarly G_{n-k} is bounded above for all $n - k$ by a suitable constant times its established asymptotic value. Also G_n is bounded below by some positive constant times its asymptotic value for sufficiently large n. Consequently

$$\begin{aligned}\sum_{k=1}^{n}\binom{2n}{2k}\frac{B_k G_{n-k}}{G_n} &= o(1)\sum_{k=1}^{n}\left(\frac{2e}{n}\right)^{k/3}c^k\\ &= o(n^{-1/3}) = o(1).\end{aligned}$$

This establishes (2.8) and hence the main theorem stated in the introduction.

3 Direct application of inclusion and exclusion

In this section we apply the method of inclusion and exclusion to count claw-free cubic graphs. We begin with some new notation. Let $cfc(k,s,t)$ be the number of claw-free cubic graphs with k components isomorphic to K_4, s diamonds and t other triangles. Then the number of nodes is

$$2n = 4k + 4s + 3t,$$

and we define

$$2m = 4s + 3t.$$

Clearly

$$cfc(k,s,t) = \frac{1}{k!}\binom{2n}{4,\ldots,4,2n-4k}cfc(0,s,t),$$

and so we focus on $cfc(0,s,t)$. The number of ways in which labels can be chosen for the s diamonds and t other triangles is

$$\frac{\binom{2m}{4,\ldots,4,3,\ldots,3}}{s!t!}\binom{4}{2}^s = \frac{(2m)!}{4^s s! 6^t t!}.$$

Now we must connect the $2s+3t = 2m-2s$ nodes of degree 2 together using $m-s$ edges. The number of ways to do this is

$$\frac{(2m-2s)!}{2^{m-s}(m-s)!}$$

but some are forbidden. We are not permitted to add an edge between two nodes of a triangle, because a multiple edge would result. And we cannot join two nodes in a diamond without creating a K_4. If i is the number of joins of a triangle to itself, then there are $\binom{t}{i}3^i$ ways to do this. If j is the number of joins of a diamond to itself, then there are $\binom{s}{j}$ ways for this to happen.

For a particular $i+j$ bad joins, the number of ways to place the remaining edges is

$$\frac{(2m-2s-2i-2j)!}{2^{m-s-i-j}(m-s-i-j)!}.$$

Hence by inclusion and exclusion

$$\begin{aligned} cfc(0,s,t) &= \left\{ \sum_{i=0}^{t}\sum_{j=0}^{s}(-1)^{i+j}3^i \binom{t}{i}\binom{s}{j}\frac{(2m-2s-2i-2j)!}{2^{m-s-i-j}(m-s-i-j)!}\right\} \\ &\qquad \cdot \frac{(2m)!}{4^s s! 6^t t!} \\ &= \frac{(2m)!}{4^s s! 6^t t!}\frac{(2(m-s))!}{2^{m-s}(m-s)!}P(s,t) \end{aligned}$$

where

$$P(s,t) = \sum_{i=0}^{t}\sum_{j=0}^{s}(-1)^{i+j}\binom{t}{i}\binom{s}{j}6^i 2^j \frac{(m-s)_{i+j}}{(2(m-s))_{2i+2j}}. \tag{3.1}$$

Since this formula was derived by inclusion and exclusion, the Bonferroni inequalities apply. Note that $m-s \geq m/2$ and so as $m \to \infty$ for $i+j = o(\sqrt{m})$ we have

$$\frac{(m-s)_{i+j}}{(2(m-s))_{2i+2j}} = \frac{1+o(1)}{2^{2i+2j}(m-s)^{i+j}}. \tag{3.2}$$

Now it can be shown that the sum in (3.1) is dominated by the terms for which (3.2) holds. In fact we can substitute the right side of (3.2) in (3.1) and apply the binomial theorem to obtain

$$P(s,t) \sim \left(1 - \frac{3}{2(m-s)}\right)^t \left(1 - \frac{1}{2(m-s)}\right)^s.$$

As $m \to \infty$, we have both

$$t/(m-s)^2 \to 0 \text{ and } s/(m-s)^2 \to 0$$

and so

$$P(s,t) \sim \exp\left(\frac{-3t}{2(m-s)}\right)\exp\left(\frac{-s}{2(m-s)}\right) = e^{-1+s/(2m-2s)}.$$

We summarize these results as follows.

Theorem 3.1. With $2n = 4k + 2m$, $2m = 4s + 3t$ and $m \to \infty$

$$cfc(k,s,t) \sim \frac{(2n)!}{(4!)^k(2n-4k)!k!}\frac{(2m)!}{2^{m-s}(m-s)!}\frac{(2m-2s)!}{4^s s! 6^t t!}e^{-\frac{2m-3s}{2m-2s}}$$

uniformly over the variables k, s and t.

Now let $cfc(m)$ denote the total number of claw-free cubics with $2m$ nodes and no components of order 4. Then

$$cfc(m) = \sum cfc(0,s,t)$$

where the sum is over all solutions of $2m = 4s + 3t$. In applying the theorem to obtain an asymptotic estimate of $cfc(m)$, it is convenient to assume that $3|m$ and hence $3|s$. The same asymptotic evaluation of $cfc(m)$ can be obtained for other values of m.

Since the approximation in the theorem is uniform in s and t we have

$$cfc(m) \sim \frac{(2m)!}{2^m}\sum_{3|s}\frac{1}{2^s}\frac{(2m-2s)!}{(m-s)!s!6^t t!}e^{-\frac{2m-3s}{2m-2s}}. \tag{3.3}$$

The sum on the right side of (3.3) is dominated by the values of s for which $s = o(\sqrt{m})$. For these values of s, we also have $t \to \infty$ and so Stirling's formula can be applied to show

$$t! = (1+o(1))\sqrt{2\pi}\sqrt{\frac{2m}{3}}\left(\frac{2m}{3e}\right)^{2m/3}\left(\frac{2m}{3}\right)^{-4s/3}$$

and

$$(m-s)^{m-s} = (1+o(1))\frac{m^{m-s}}{e^s}.$$

After some simplification we find that the contribution of the terms for which $s = o(\sqrt{m})$ to the right side of (3.3) is

$$(1+o(1))\frac{(2m)!}{e\sqrt{6\pi m}}\left(\frac{m}{2e}\right)^{m/3} 3\sum_{3|s}\frac{((m/2)^{1/3})^s}{s!}$$

which leads to the same estimate for H_n in the main theorem. As for the terms omitted, Stirling's formula still serves to obtain a suitable upper bound on this remainder to show that they are negligible.

4 Connectivity of claw-free cubics

Let C_n be the number of labeled connected claw-free cubic graphs of order $2n$. Then H_n and C_n are related by the following well-known relation (compare (2.7)):

$$H_n = \sum_{k=2}^{n}\binom{2n}{2k}\frac{k}{n}C_k H_{n-k}.$$

Therefore, to show that almost all *cfc*'s are connected, i.e. $H_n \sim C_n$, we need only show that

$$\sum_{k=2}^{n/2}\binom{2n}{2k}H_k H_{n-k}/H_n = o(1). \tag{4.1}$$

Using the formula (1.3) of our main theorem, Stirling's formula and simple estimates, we find that the left side of (4.1) is

$$O(1)\sum_{k=2}^{n/2}\frac{1}{\sqrt{k}}\left(\frac{k}{n-k}\cdot\exp\{-1+3/\sqrt[3]{2k^2}\}\right)^{k/3}.$$

Now this sum is split in two parts according as $k \le \log n$ or $k > \log n$. Then it can be shown that for $2 \le k \le \log n$, the value of the sum is $O(n^{-2/3+\varepsilon})$ and for $\log n < k \le n/2$ it is $O(n^{-1/3+\varepsilon})$, where $\varepsilon > 0$ is arbitrary. Hence (4.1) is satisfied. We could also establish this using the fact that almost all *cfc*'s are derived from general cubic graphs which are in turn almost surely connected.

Turning to 2-connectivity, we have already seen that *cfc*'s derived from general cubics with one or more loops are negligible in number. A loopless general cubic graph is a cubic multigraph. It was shown in [RoW01] that almost surely such multigraphs are hamiltonian, hence 2-connected. Substituting strings of diamonds for edges preserves 2-connectivity, hence *cfc*'s almost surely have vertex connectivity $\kappa \geq 2$. Since they have about $(n/2)^{1/3}$ diamonds, we know also that $\kappa < 3$ almost surely.

Corollary 4.1 Almost all claw-free cubic graphs have vertex connectivity $\kappa = 2$.

This result is in contrast with cubic graphs, which have $\kappa = 3$ almost surely [Wo79].

5 Conclusion

Our asymptotic estimate can be compared with exact values calculated from (1.1). For $n = 26$, the number H_{26} of *cfc*'s of order 52 is $10.1603149 \times 10^{72}$ (see (1.2)). On the other hand, the estimate from our theorem is 10.931×10^{72}, which is high by 7.59%. However more extensive calculation is reassuring. The data indicates that convergence of the asymptotic estimate to the exact value of H_n is steady (the relative error positive but monitonically decreasing) for $n \geq 1172$ and the relative error is of order $O(n^{-1/3})$. By order 10000 we find that $H_{5000} = 6.6655 \times 10^{40601}$ and the asymptotic estimate is high by 2.92%.

There is another check that can be made. Suppose we assume that for some constants $c > 0$ and $a > 0$

$$H_{n+1}/H_{n-1} \sim cn^a.$$

Then for fixed k

$$H_{n-k}/H_{n-1} \sim 1/(\sqrt{c}n^{a/2})^{k-1}.$$

Now on dividing both sides of the recurrence relation (1.1) by H_{n-1} and combining terms we can examine the exponents of the positive terms and find that the only possible solution comes from the second term, which shows that we must have

$$cn^a \sim \frac{2^5}{\sqrt{c}} n^{7-a/2}.$$

Hence

$$\frac{H_{n+1}}{H_{n-1}} \sim (2^5 n^7)^{2/3},$$

a result which also follows from formula (1.3) of our main theorem.

It also follows easily from our theorem and the formula ((2.3) above with $\ell = d = 0$) of the second author [Re59] for the asymptotic number of cubic graphs that almost all cubic graphs have claws.

We close with a corollary on the number of unlabeled *cfc*'s.

Corollary 5.1. The number U_n of unlabeled claw-free cubic graphs of order $2n$ has the asymptotic value:

$$U_n \sim \frac{1}{\sqrt{6\pi n}} \left(\frac{n}{2e}\right)^{n/3} e^{2(n/2)^{1/3}}.$$

Proof. For any graph G, labeled or not, let $s(G)$ be the number of automorphisms of G. Since the number of ways to label a graph of order $2n$ is $(2n)!/s(G)$,

$$U_n = \sum s(G)/(2n)!$$

where the sum is over all labeled *cfc*'s of order $2n$. We denote the number of diamonds in a *cfc* by $j(G)$. Then

$$s(G) \geq 2^{j(G)}$$

and

$$U_n \geq \sum 2^{j(G)}/(2n)!\ . \tag{5.1}$$

The right side of this inequality can be estimated by using formula (2.4) without the factor of $(2n)!$ and the last factor $\left(\frac{1}{2}\right)^j$. Now the sum over j in the modified version of the right side of (2.4) peaks near $j = 2(n/2)^{1/3}$. On summing over ℓ and d as in Section 2 the result is the asymptotic estimate

$$\sum 2^{j(G)}/(2n)! \sim \frac{1}{\sqrt{6\pi n}} \left(\frac{n}{2e}\right)^{n/3} e^{2(\frac{n}{2})^{1/3}}.$$

This gives the desired asymptotic lower bound for U_n.

To finish the proof we need to show that the contributions of all the cases in which $s(G)$ is strictly greater than $2^{j(G)}$ are negligible. This can be established using the fact that *cfc*'s are derived from general cubic graphs, which almost surely have the identity automorphism group. Actually we need that

$$\sum 2^{j(G)} = o(H_n)$$

where the sum is over all labeled *cfc*'s of order $2n$ which have non-trivial automorphism groups. This can be verified by combining the methods leading to Cor. 3.12 in [McW84] with those leading to Cor. 1 in [W81]. □

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