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COUNTING CLAW-FREE CUBIC GRAPHS*

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Abstract. Let H_n be the number of claw-free cubic graphs on $2n$ labeled nodes. Combinatorial reductions are used to derive a second order, linear homogeneous differential equation with polynomial coefficients whose power series solution is the exponential generating function for $\{H_n\}$. This leads to a recurrence relation for H_n which shows $\{H_n\}$ to be P-recursive and which enables the sequence to be computed efficiently. Thus the enumeration of labeled claw-free cubic graphs can be added to the handful of known counting problems for regular graphs with restrictions which have been proved P-recursive.

Key words. labeled graph counting, claw-free graph, cubic graph, exponential generating function, P-recursive sequence

AMS subject classifications. 05A15, 05C30

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1. Introduction.

The problem of generating cubic graphs, i.e., 3-regular graphs, has been studied for over 100 years using combinatorial reductions [6]. Read applied combinatorial reductions to the derivation of an efficient recurrence relation for counting the number of labeled connected cubic graphs on $2n$ nodes [12], in which the nodes are labeled but not the edges. He observed that expressing the recurrence relations in terms of an exponential generating function (EGF) resulted in substantial simplifications. This allowed him to derive a second order linear differential equation for the EGF of all labeled cubic graphs (not necessarily connected). Later, Wormald [16] incorporated EGFs directly into the reduction approach in order to obtain differential equations for the EGFs of cubic graphs of given k-connectivity ($k = 0$, 1, 2, and 3). He derived recurrence relations only at the end of the process. In the present paper we will follow this pattern in deriving a recurrence relation for the exact number H_n of labeled claw-free cubic graphs on $2n$ nodes. A graph is *claw-free* if and only if it contains no induced subgraph isomorphic to $K_{1,3}$. In a cubic graph, this is equivalent to the condition that every vertex lies on a triangle, i.e., on a 3-cycle.

Claw-free graphs have been studied in relation to independent sets, perfect graphs, Hamiltonicity, reconstruction, and matchings. References may be found in the introduction of [10]. In particular, claw-free graphs which are 3-regular or 4-regular have been amenable to analysis of extendibility of matchings [9]. Related questions and conjectures on Hamiltonicity arising from this work are presented in [11]. For cubic claw-free graphs, Plummer asks for the probabilistic behavior of Hamiltonicity in cubic claw-free graphs, in the planar case, and in general. The latter question was answered in [13] where it was determined that almost all claw-free cubic graphs are Hamiltonian. For 4-connected 4-regular claw-free graphs, Plummer conjectures that all are Hamiltonian [11, Conjecture 3.8]. The asymptotic behavior of the sequence

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$\{H_n\}$ was determined in [8], the results of which were essential for the Hamiltonicity result of [13] cited above. The enumeration of claw-free cubic graphs with given connectivity is treated in [1]. The method requires enumeration results for labeled general cubic graphs [2].

The recurrence relation obtained in section 3 for H_n allows the n numbers $H_1, \dots,$ H_n to be calculated with $O(n)$ arithmetic operations. It is a linear homogeneous recurrence of order 12 in which the coefficients are polynomials in n. These polynomials range in degree up to 23 and all have integer coefficients. This recurrence shows that $\{H_n\}$ belongs to the class of P-recursive sequences, first defined by Stanley [14]. It was later shown that a number of labeled graph enumeration problems, including cubic graphs, are P-recursive [5]. Gessel generalized those results considerably and showed that for any fixed r the number of labeled r-regular graphs is P-recursive [4]. However, Gessell commented on the lack of general methods for proving P-recursiveness of the number of r-regular graphs subject to restrictions on connectivity, girth, and the like. For restricted labeled cubic graphs there are two examples of P-recursive counting problems provided by Wormald; those rooted at a triangle [15], and those containing no triangle [17]. To these we can now add the enumeration of labeled claw-free cubic graphs. However, for labeled cubic and claw-free cubic graphs which are k-connected for $k = 1$, 2, or 3 the question of P-recursiveness is open, as the enumerations provided in [16] and [1] do not provide linear recurrences.

For general graph theoretic terminology and notation we follow [7], except for adopting the more modern names nodes and edges in place of points and lines. In particular, we assume a basic knowledge of labeled enumeration techniques using EGFs, such as is provided by Chapter 1 of [7].

2. Structural properties. All graphs to be considered will have nodes labeled but not edges. A *claw-free* graph is one with no induced subgraph isomorphic to $K_{1,3}$. We will deal exclusively with *cubic* graphs, i.e., 3-regular graphs. For cubic graphs, the claw-free condition is equivalent to requiring that every node should belong to a triangle. We will count the number H_n of labeled claw-free cubic graphs on $2n$ nodes.

In any cubic graph, the maximum number of triangles in which a node may lie is 3, and this can occur only in a component isomorphic to K_4. In our counting, we will account for such components at the end. A node may lie in exactly two triangles precisely if it is one of the nodes of degree 3 in an induced subgraph isomorphic to $K_4 - e$; we call such a subgraph a *diamond*. A maximal set of diamonds which are adjacent in series is called a *string of diamonds*. A connected graph in which every node is contained in a diamond is called a *ring of diamonds*. For the purposes of counting, we consider a single edge to be a trivial string of diamonds, provided it is not incident to a diamond. However, a ring of diamonds must contain at least two diamonds. Like copies of K_4, rings of diamonds will be accounted for explicitly at the end of the process.

Denote by *reduction* the operation of replacing each string of diamonds by a single edge. For any claw-free cubic graph with no component isomorphic to K_4 or a ring of diamonds, the reduced graph must be a cubic multigraph in which every node is contained in exactly one triangle (defined as a set of three mutually adjacent nodes). Clearly, none of the edges in these remaining disjoint triangles resulted from the reduction of a nontrivial string of diamonds unless it belongs to a double edge, the nodes of which are mutually adjacent to a third node. Such a configuration is termed a *trumpet*. In the double edge of a trumpet, exactly one of the two edges must have resulted from the reduction of a nontrivial string of diamonds. Since our edges are not labeled, for counting purposes it does not matter which edge is which in the double. Denote by *expansion* the operation which is inverse to reduction.

Now, in a reduced graph we can contract each of the disjoint triangles to a single node; denote this operation by *contraction* and its inverse by *dilation*. The contraction of a trumpet will be a loop. The contraction of a reduced graph is an arbitrary cubic general graph. We could, if we wished, contract an unreduced graph by contracting just those triangles which do not overlap any other triangle. Then reduction and contraction are easily seen to be commutative operations.

The approach that we will take to counting claw-free cubic graphs is to start with cubic general graphs, dilate and expand them, and then add in components isomorphic to K_4 or a ring of diamonds.

3. Labeled cubic general graphs. Let $g_{s,d,l}$ be the number of labeled cubic general graphs without triple edges having exactly s single edges, d double edges, and l loops. Note that the number $2n$ of nodes is just

$$2n = \frac{2s + 4d + 2l}{3} .$$

It is the nodes that are labeled. Also, trumpets are not distinguished from other double edges in this treatment. The graphs are not necessarily connected, so we let $g_{0,0,0} = 1$.

Now let $G(x, y, w)$ be the exponential generating function

$$G(x, y, w) = \sum_{s,d,l} g_{s,d,l}\, x^s y^d w^l / (2n)! .$$

The partial derivations with respect to x, y, and w will be denoted G_x, G_y, and G_w, and similarly for higher order derivatives. Clearly G_x is the exponential generating function for labeled cubic general graphs without triple edges which are rooted at a single edge, except that the root edge is not represented by a factor of x. The other first order partial derivatives have like interpretations, as do the higher order derivatives. To derive an expression for G_x, we can imagine removing a single edge from a general cubic graph, leaving two nodes of degree 2. These are then smoothed over, leaving edges which we think of as root edges. The possibilities for the latter are counted by appropriate partial derivatives of G, in general, depending upon whether the root edges are singles, doubles, triples, ordinary loops, or nodeless loops. The latter occurs when an edge incident to a loop is removed. One must also multiply by a monomial which accounts for the various edges which were deleted after the original root edge was removed.

If a cubic graph is originally rooted at a single edge, then after deleting the root we have 17 possibilities for the two new root edges, as shown in Table 1 along with the corresponding exponential generating function.

Hence we have

$$\begin{aligned} G_x &= \left(\frac{w^2}{2} + \frac{x^5}{4} + \frac{x^2yw}{2} + \frac{x^4y^2}{8}\right) G + \left(x^2w + \frac{x^4y}{2}\right) G_x \\ &\quad + \left(\frac{x^4}{2} + x^3w + \frac{x^5y}{2}\right) G_y + \left(yw + \frac{x^2y^2}{2}\right) G_w \\ &\quad + \frac{x^4}{2}G_{xx} + x^5G_{xy} + x^2yG_{xw} + \frac{x^6}{2}G_{yy} + x^3yG_{yw} + \frac{y^2}{2}G_{ww} . \end{aligned} \tag{1}$$

TABLE 1
Terms contributing to G_x.

EGF	Root edges
$\frac{w^2}{2}G$	two nodeless loops
$\frac{x^5}{4}G$	belong to same triple edge
$\frac{x^2yw}{2}G$	triple edge and nodeless loop
$\frac{x^4y^2}{8}G$	two triple edges
x^2wG_x	single edge and nodeless loop
$\frac{x^4y}{2}G_x$	single edge and triple edge
$\frac{x^4}{2}G_y$	belong to same double edge
x^3wG_y	double edge and nodeless loop
$\frac{x^5y}{2}G_y$	double edge and triple edge
ywG_w	ordinary loop and nodeless loop
$\frac{x^2y^2}{2}G_w$	ordinary loop and triple edge
$\frac{x^4}{2}G_{xx}$	two single edges
x^5G_{xy}	single edge and double edge
x^2yG_{xw}	single edge and ordinary loop
$\frac{x^6}{2}G_{yy}$	two double edges
x^3yG_{yw}	ordinary loop and double edge
$\frac{y^2}{2}G_{ww}$	two ordinary loops

If we wished a recurrence relation capable of determining all of the numbers $g_{s,d,l}$ starting with the initial condition $g_{0,0,0} = 1$, we would need only extract the coefficient of $x^{s-1}y^dw^l$ from both sides of (1) and set the values equal. This is because every nonempty cubic general graph without triple edges must contain at least one single edge. However, to compute the numbers corresponding to all such graphs on up to $2n$ nodes by way of this recurrence would require $O(n^3)$ arithmetic operations. As we shall see, the number of claw-free cubic graphs is P-recursive as a function of n and can therefore be calculated in $O(n)$ operations. This will require the use of separate equations for G_y, G_w, and each of the second order partial derivatives except for G_{xx}.

To obtain an equation for G_y similar to (1) for G_x, consider a cubic general graph rooted at a double edge. We then remove the double edge and splice the two edges which were adjacent to the root together into a new edge which we designate as the root for the reduced graph. The latter cannot form a nodeless loop, since the original root was not part of a triple edge. However, it can belong to a triple edge, be a single edge, belong to a double edge, or be an ordinary loop. These possibilities give, in order, the four terms on the right side of the next equation:

$$G_y = \frac{x^2y}{2}G + x^2G_x + x^3G_y + \frac{x^2}{2}G_w \,. \tag{2}$$

Finally, a cubic general graph rooted at a loop can be reduced by removing the loop and its adjacent edge. This leaves a vertex of degree 2, which we remove and splice the two incident edges into a new edge. The latter becomes the root of the reduced graph; the root can be a nodeless loop, belong to a triple edge, be a single edge, belong to a double edge, or be a loop. These possibilities correspond in that order to the five terms on the right side of this equation:

$$G_w = \left(xw + \frac{x^3y}{2}\right)G + x^3G_x + x^4G_y + xyG_w \ . \tag{3}$$

Finally, the differentiation of (2) and (3) with respect to x, y, and w is straightforward. Making use of the fact that the order of differentiation is immaterial, we obtain the following equations for the second order partial derivatives:

$$G_{yw} = \frac{x^2y}{2}G_w + x^2G_{xw} + x^3G_{yw} + \frac{x^2}{2}G_{ww} \ , \tag{4}$$

$$G_{xy} = xyG + \left(2x + \frac{x^2y}{2}\right)G_x + 3x^2G_y + xG_w + x^2G_{xx} + x^3G_{xy} + \frac{x^2}{2}G_{xw} \ , \tag{5}$$

$$\begin{aligned} G_{xw} = {} & \left(w + \frac{3x^2y}{2}\right)G + \left(3x^2 + xw + \frac{x^3y}{2}\right)G_x + 4x^3G_y + yG_w \\ & + x^3G_{xx} + x^4G_{xy} + xyG_{xw} \ , \end{aligned} \tag{6}$$

$$G_{yy} = \frac{x^2}{2}G + \frac{x^2y}{2}G_y + x^2G_{xy} + x^3G_{yy} + \frac{x^2}{2}G_{yw} \ , \tag{7}$$

$$G_{ww} = xG + \left(xw + \frac{x^3y}{2}\right)G_w + x^3G_{xw} + x^4G_{yw} + xyG_{ww} \ . \tag{8}$$

4. Claw-free cubic graphs. Let $H(z^2)$ be the exponential generating function for counting all labeled claw-free cubic graphs so that

$$H(z) = \sum_{n=0}^{\infty} \frac{H_n z^n}{(2n)!} \ .$$

Our objective is to derive a linear, homogeneous differential equation with coefficients rational in z which is satisfied by $H(z)$. This will imply that the coefficients form a P-recursive sequence, and hence that the n numbers $H_1, \ldots, H_n$ can be calculated in $O(n)$ operations.

The major portion of $H(z)$ is accounted for by the expansion and dilation of the triple-edge-free general cubic graphs counted by $G(x, y, w)$. The strings of diamonds which can reduce to a single edge are counted by

$$b(z) = (1 - z^2/2)^{-1} \ . \tag{9}$$

We leave $b = b(z)$ unexpanded as long as possible in order to simplify our equations. Then to count the graphs resulting from expansion and dilation we simply perform the substitutions

$$\begin{aligned} x &= zb \ , \\ y &= \frac{z^2b^2}{2} \ , \\ w &= \frac{z^3b}{4} \ . \end{aligned} \tag{10}$$

Note that after substitution of z^2 for z in the formula for w, the very first term is $180\frac{z^6}{6!}$. The exponent of z counts the two vertices of the trumpet horn and the four of the mandatory diamond. Since there are four automorphisms, the coefficient is $\frac{6!}{4} = 180$.

Now $G(z^2)$ counts everything in $H(z^2)$ except for components isomorphic to K_4 or a ring of diamonds, or which reduce to a triangular prism (since that contracts to a triple edge). These are counted, respectively, by $z^2/24$, $-z^2/4 + \ln(\sqrt{b})$, and $z^3b^3/12$. The second of these may require explanation; a ring of m diamonds has $2m2^m$ automorphisms, so the counting series for these components is

$$\sum_{m=2}^{\infty} \frac{z^{2m}}{2m2^m} = -\frac{z^2}{4} - \frac{1}{2}\ln(1 - z^2/2)\,.$$

We then exponentiate to count all graphs consisting entirely of components of these three types. Let $\varphi(z^2)$ be the resulting exponential generating function. Then

$$\varphi(z) = \sqrt{b}\exp\left(-\frac{5z^2}{24} + \frac{z^3b^3}{12}\right) \tag{11}$$

and

$$H(z) = \varphi(z)G(z)\,. \tag{12}$$

The differential equation satisfied by $H(z)$ is now determined by the set of equations (1)–(12). From (11) and (12) we have

$$\begin{aligned} H'(z) = {}& \frac{\varphi'(z)}{\varphi(z)} \cdot \varphi(z)G(z) + x'(z) \cdot \varphi(z)G_x(z) \\ & + y'(z) \cdot \varphi(z)G_y(z) + w'(z) \cdot \varphi(z)G_w(z)\,. \end{aligned} \tag{13}$$

Differentiating again with respect to z we find

$$\begin{aligned} H''(z) = {}& \frac{\varphi''(z)}{\varphi(z)} \cdot \varphi(z)G(z) + 2x'(z)\frac{\varphi'(z)}{\varphi(z)} \cdot \varphi(z)G_x(z) \\ & + 2y'(z)\frac{\varphi'(z)}{\varphi(z)} \cdot \varphi(z)G_y(z) + 2w'(z)\frac{\varphi'(z)}{\varphi(z)} \cdot \varphi(z)G_w(z) \\ & + x''(z) \cdot \varphi(z)G_x(z) + y''(z) \cdot \varphi(z)G_y(z) + w''(z) \cdot \varphi(z)G_w(z) \\ & + x'(z)^2 \cdot \varphi(z)G_{xx}(z) + 2x'(z)y'(z) \cdot \varphi(z)G_{xy}(z) \\ & + 2x'(z)w'(z) \cdot \varphi(z)G_{xw}(z) + y'(z)^2 \cdot \varphi(z)G_{yy}(z) \\ & + 2y'(z)w'(z) \cdot \varphi(z)G_{yw}(z) + w'(z)^2 \cdot \varphi(z)G_{ww}(z)\,. \end{aligned} \tag{14}$$

We now consider (13) and (14) as linear equations in the 12 unknown quantities $H''(z)$, $H'(z)$, $H(z) = \varphi(z)G(z)$, $\varphi(z)G_x(z)$, $\varphi(z)G_y(z)$, $\varphi(z)G_w(z)$, $\varphi(z)G_{xx}(z)$, $\varphi(z)G_{xy}(z)$, $\varphi(z)G_{xw}(z)$, $\varphi(z)G_{yy}(z)$, $\varphi(z)G_{yw}(z)$, and $\varphi(z)G_{ww}(z)$. The coefficients are polynomials in z and b. To see this, note that $b'(z) = zb^2(z)$. Thus all derivatives of x, y, and w can be expressed as polynomials in z and b. Moreover the ratios $\varphi'(z)/\varphi(z)$ and $\varphi''(z)/\varphi(z)$ are also polynomials in z and b. Equations (1)–(8) can all be converted to the same format by applying the substitutions in (10) and multiplying through by $\varphi(z)$. Thus we have 10 linear equations in these 12 unknowns. With the help of the symbolic Gaussian elimination procedure in Maple [3], we can eliminate all of the unknown quantities except for $H(z)$, $H'(z)$, and $H''(z)$. This leads to the linear differential equation

$$\begin{aligned}
0 &= (144z^8 + 288z^7 - 576z^4)H''(z) \\
&\quad + (-36z^{10} - 96z^9 + 24z^8 + 144z^7 + 576z^6 + 384z^5 \\
&\qquad - 576z^4 - 2880z^3 - 576z^2 + 1152)H'(z) \\
&\quad + (-15z^{11} - 74z^{10} - 130z^9 - 96z^8 + 144z^7 + 368z^6 + 336z^5 - 288z^4 \\
&\qquad - 240z^3 - 288z^2 - 96z)H(z)\,.
\end{aligned} \tag{15}$$

Here the substitution (9) has been applied to express the coefficients as rational functions of z, common factors have been removed from the three coefficients, and they have been multiplied by a suitable polynomial so that the three coefficients have all become polynomials in z with integer coefficients.

The power series $H(z)$ is the Taylor series about $z = 0$ of the unique solution to (15) which satisfies the initial conditions $H(0) = 1$ and $H'(0) = 0$.

A recurrence relation for the coefficients of $H(z)$ is obtained by extracting the coefficient of $z^n/(2n)!$ in (15); this must be equal to 0. The term $1152H'(z)$ contributes $H_{n+1}/(4n+2)$. This has the maximum index in H, so we solve for H_{n+1} by equating it to $-(2n+1)/576$ times the sum of the other terms. In general, the term contributed by $2^k H(z)$ is $\binom{2n}{2k}(2k)!H_{n-k}$, which upon multiplying by $(2n+1)$ becomes $\binom{2n+1}{2k+1}(2k+1)!H_{n-k}$. For $k \geq 1$, the term contributed by $(2n+1)z^k H'(z)$ is $(n-k+1)\binom{2n+1}{2k+1}(2k+1)!H_{n-k+1}$. Finally, for $k \geq 2$ the term contributed by $(2n+1)z^k H''(z)$ is $(n-k+2)(n-k+1)\binom{2n+1}{2k+1}(2k+1)!H_{n-k+2}$. In this way we find the following relation, which is valid for $n \geq 1$:

$$\begin{aligned}
H_{n+1} &= (6n-5)\binom{2n+1}{3}H_{n-1} + 60(2n^2-7)\binom{2n+1}{5}H_{n-2} \\
&\quad + 420(12n-31)\binom{2n+1}{7}H_{n-3} - 60480(4n-19)\binom{2n+1}{9}H_{n-4} \\
&\quad - 3326400(6n^2-54n+127)\binom{2n+1}{11}H_{n-5} \\
&\quad - 172972800(9n^2-108n+347)\binom{2n+1}{13}H_{n-6} \\
&\quad - 54486432000(n-1)\binom{2n+1}{15}H_{n-7} \\
&\quad + 59281238016000(n-7)\binom{2n+1}{17}H_{n-8} \\
&\quad + 422378820864000(18n-97)\binom{2n+1}{19}H_{n-9} \\
&\quad + 6563766876226560000\binom{2n+1}{21}H_{n-10} \\
&\quad + 67322960257512960000\binom{2n+1}{23}H_{n-11}\,.
\end{aligned} \tag{16}$$

Of course H_{n-j} is zero whenever $j > n$. With the initial conditions $H_0 = 1$ and $H_1 = 0$, (16) can be used to compute the values of $H_2, \ldots, H_{n+1}$ using just $O(n)$ arithmetic operations. In this way we computed the values shown in Table 2.

TABLE 2
Numbers of labeled cubic claw-free graphs.

H_n	n
1	2
60	3
2555	4
466200	5
62791575	6
14536021500	7
8381453705625	8
3284480337138000	9
1942832950684250625	10
2143745512307546647500	11
1743194710893176557891875	12
2022583790860881671548125000	13
3687297941048128552947911484375	14
5250396961636474882113432240187500	15
10270576798318031167485848746426640625	16
28247581137945084450497132391551830500000	17
63409618548369444745423852264233423897890625	18
189787893059957073451746036716319750214365937500	19
739731302424534941124199455315845613980976141796875	20
2436293022465856848407798760164672100623479345846875000	21
10433013033263780019056740194457690414996014419582021484375	22
55053013693844064927863480169144644331902982938883731835937500	23
252448493699621454815261719991354533831171674212674184547416015625	24
1472749695048011678818262827491781703308289147738221578121708593750000	25
10160314924243373000701474995668144304893902876648285295864422890087890625	26

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