Hamilton Cycles Containing Randomly Selected Edges in Random Regular Graphs*

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Abstract

In previous papers the authors showed that almost all d-regular graphs for $d \geq 3$ are hamiltonian. In the present paper this result is generalized so that a set of j oriented root edges have been randomly specified for the cycle to contain. The Hamilton cycle must be orientable to agree with all of the orientations on the j root edges. It is shown that the requisite Hamilton cycle almost surely exists if $j = o(\sqrt{n})$, and the limiting probability distribution at the threshold $j = c\sqrt{n}$ is determined when $d = 3$. It is a corollary (in view of results elsewhere) that almost all claw-free cubic graphs are hamiltonian.

There is a variation in which an additional cyclic ordering on the root edges is imposed which must also agree with their ordering on the Hamilton cycle. In this case, the required Hamilton cycle almost surely exists if $j = o(n^{2/5})$.

The method of analysis is small subgraph conditioning. This gives results on contiguity and the distribution of the number of Hamilton cycles which imply the facts above.

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1 Introduction

We showed in [12] and [13] that for fixed $d \geq 3$, almost all labelled d-regular graphs are hamiltonian (as the number of vertices tends to infinity). In this paper we are concerned with a strong improvement of this result. We show in particular that for a random d-regular graph with $o(\sqrt{n})$ edges randomly specified, there is asymptotically almost surely a Hamilton cycle containing all the specified edges.

We will be concerned with the set of labelled d-regular graphs on n vertices with j distinguished edges called *root edges* each of which is provided with an orientation. (Throughout this paper, we assume that n takes on only even values if d is odd.) This will be regarded as a probability space, $\Gamma_{n,d}^{(j)}$, with the uniform distribution. A cycle containing the root edges in a graph is said to *respect* their orientation if one of the two orientations of the cycle agrees with the orientation of every root edge. For $G \in \Gamma_{n,d}^{(j)}$, let $\mathcal{H}$ be the event that G contains a Hamilton cycle including all j of the root edges and respecting their orientations.

Theorem 1 *For $G \in \Gamma_{n,3}^{(j)}$*

$$\mathbf{P}(\mathcal{H}) = e^{-2j^2/3n} + o(1)$$

uniformly over all $j = j(n)$.

The corresponding result for $j = 0$ was proved in [12]. When two oriented root edges both point towards or both point away from a common incident vertex, then the event $\mathcal{H}$ is impossible. This is a rare event until j reaches about $\sqrt{n}$, and in fact the limiting distribution expressed in Theorem 1 is asymptotically the same as for this event. Hence for $j = O(\sqrt{n})$, the probability of the event $\mathcal{H}$, conditioned on the event that no such pairs of oriented root edges occur, tends to 1.

Our interest in this work originated from the following application. By *asymptotically almost surely* (a.a.s.) we mean occurring with probability tending to 1 as $n \to \infty$.

Corollary 1 *A random claw-free cubic graph is a.a.s. hamiltonian.*

We will also consider a variation in which the root edges are cyclically ordered. For $j \geq 1$, the set of labelled d-regular graphs on n vertices with j oriented root edges and also with a cyclic ordering imposed on the root edges determines a uniform probability space which we call $\Gamma_{n,d}^{<j>}$. We say that the cyclic ordering is *respected* by a cycle containing the oriented root edges if it is the same as the cyclic order of the root edges induced by that order of traversal by the cycle which agrees with the orientations of the root edges.

Theorem 2 *Let $j = o(n^{2/5})$. Then $G \in \Gamma_{n,3}^{<j>}$ a.a.s. has a Hamilton cycle through all the root edges respecting both their cyclic ordering and their orientations.*

We also obtain analogues (and a slight generalization) of the first two theorems for d-regular graphs. Define $\Gamma_{n,d}^{(j)\setminus\{m\}}$ to be the uniform probability space of d-regular graphs on n vertices with j oriented root edges, and with m other edges marked. (Marked edges will be forbidden from a Hamilton cycle.) Also define $\Gamma_{n,d}^{<j>\setminus\{m\}}$ similarly with a cyclic orientation of the root edges given.

Theorem 3 *Let $d \geq 3$.*

(i) For $j = o(\sqrt{n})$ and $m = o(\sqrt{n})$, $G \in \Gamma_{n,d}^{(j)\setminus\{m\}}$ a.a.s. has a Hamilton cycle through all the root edges respecting their orientations, and avoiding all marked edges.

(ii) For $j = o(n^{2/5})$ and $m = o(n^{1/2})$, $G \in \Gamma_{n,d}^{<j>\setminus\{m\}}$ a.a.s. has a Hamilton cycle through all the root edges respecting both their cyclic ordering and their orientations, and avoiding all marked edges.

To prove these theorems we will use the basic method in [12] and [13], in which we analyze the variance of the number of Hamilton cycles of the required type. In ordinary regular graphs, the variance is "explained" asymptotically by the way the numbers of short cycles in a graph influence the expected number of Hamilton cycles. For this particular problem there is an interesting variation: the numbers of pairs of root edges of short distance apart also affects the expected number of cycles passing through all root edges and conforming to the required conventions.

One can presumably sharpen Theorem 3(i) so as to obtain the type of limiting distribution for the probability given in Theorem 1, however this would require a rather more complicated variance computation, much as the variance computation in Frieze et al. [6, Section 4] for $d \geq 4$ is more complicated than the case $d = 3$ in [11].

Note that by forgetting the orientations of the root edges in Theorems 1 and 3(i), we obtain lower bounds on the probability $\mathcal{H}'$ that a randomly selected set of j edges of a random d-regular graph is contained in some Hamilton cycle of the graph. So for $j = o(\sqrt{n})$ this probability tends to 1. Pairs of adjacent unoriented root edges cause no problem for Hamilton cycles, so there is no local configuration expected to prevent hamiltonicity until j is approximately $n^{2/3}$, at which point the event B that three root edges are incident with the same vertex is no longer rare. Trivially, $\mathbf{P}(\mathcal{H}') \leq \mathbf{P}(B)$.

Conjecture 1 *If $O(n^{2/3})$ edges are randomly specified in a random d-regular graph, then $\mathbf{P}(\mathcal{H}') = \mathbf{P}(B) - o(1)$.*

This conjecture, if true, gives the limit of $\mathbf{P}(\mathcal{H}')$ as a function of $jn^{-2/3}$. In particular, it implies that a randomly selected set of $j = o(n^{2/3})$ edges of a random d-regular graph is a.a.s. contained in some Hamilton cycle, which is the best possible value of j if true. On the other hand, we have no good idea of the threshold j at which the conclusions in Theorem 2 and 3(ii) begin to fail, apart from the fact that it lies between $n^{2/5}$ and $\sqrt{n}$.

Theorem 3 is proved by showing two results on asymptotic equivalence. The first is that a random 3-regular graph with j root edges and m marked edges is equivalent, in a sense to be made precise, to a random Hamilton cycle with j root edges plus a random perfect matching with m marked edges. The second is that a random d-regular graph with j root edges and m marked edges is equivalent to a $d-1$-regular graph of the same type, plus a perfect matching.

To define the equivalence, suppose that $(\mathcal{G}_n)_{n\geq 1}$ and $(\bar{\mathcal{G}}_n)_{n\geq 1}$ are two sequences of probability spaces such that $\mathcal{G}_n$ and $\bar{\mathcal{G}}_n$ differ only in the probabilities. We say that these two sequences are *contiguous* if a sequence of events A_n is a.a.s. true in $\mathcal{G}_n$ if and only if it is a.a.s. true in $\bar{\mathcal{G}}_n$, in which case we write

$$\mathcal{G}_n \approx \bar{\mathcal{G}}_n.$$

Contiguity is clearly an equivalence relation on sequences of spaces.

To formalize the addition of two random graphs, we make the following definitions. If $\mathcal{G}$ and $\bar{\mathcal{G}}$ are two probability spaces of random graphs or multigraphs on the same vertex set, their *sum* $\mathcal{G}+\bar{\mathcal{G}}$ is the space whose elements are defined by the random multigraph $G \cup \bar{G}$ where $G \in \mathcal{G}$ and $\bar{G} \in \bar{\mathcal{G}}$ are generated independently, and in the union their edge sets are unioned as multisets (with additive multiplicity). The *graph-restricted sum* of $\mathcal{G}$ and $\bar{\mathcal{G}}$, denoted by $\mathcal{G} \oplus \bar{\mathcal{G}}$, is the restriction of $\mathcal{G}+\bar{\mathcal{G}}$ to graphs (with no multiple edges). It is a consequence of the work in [12] that for n even, the graph-restricted sum of a random perfect matching with a Hamilton cycle on an even number of vertices is contiguous to a random 3-regular graph, and from [13] that for n even and $d \geq 2$, $\Gamma^{(0)}_{n,d} \oplus \Gamma^{(0)}_{n,1} \approx \Gamma^{(0)}_{n,d+1}$. The theory of sums of random regular graph spaces has developed considerably from this point (see [15]). The following theorems add to this theory, besides being the key to the proof of Theorem 3.

Theorem 4 *Let $\Gamma^{(j)}_{n,\mathrm{ham}}$ denote the uniform random graph space whose elements are Hamilton cycles on n vertices with j root edges specified and oriented one of the two consistent ways along the cycle, and let $\Gamma^{\{m\}}_{n,1}$ denote the uniform random graph space whose elements are perfect matchings on n vertices with m marked edges (unoriented). Then for $j = o(\sqrt{n})$ and $m = o(\sqrt{n})$ with n even,*

$$\Gamma^{(j)\backslash\{m\}}_{n,3} \approx \Gamma^{(j)}_{n,\mathrm{ham}} \oplus \Gamma^{\{m\}}_{n,1}.$$

Theorem 5 *For $d \geq 3$, $j = o(\sqrt{n})$ and $m = o(\sqrt{n})$ with n even,*

$$\Gamma^{(j)\backslash\{m\}}_{n,d} \approx \Gamma^{(j)\backslash\{m\}}_{n,d-1} \oplus \Gamma^{(0)}_{n,1}.$$

Remark The results analogous to Theorems 4 and 5 for random regular multigraphs also hold. These are obtained essentially as intermediate results in the proofs.

We close this section with a proof of the implications regarding claw-free cubic graphs.

Proof of Corollary 1 By McKay et al. [8] it was shown that a.a.s. a labelled claw-free cubic graph F on n vertices is obtained in a unique way from a labelled cubic graph G on $m \sim n/3$ vertices with a set S of specified edges such that F is hamiltonian if, and only if, G has a Hamilton cycle containing all the edges in S. Letting $s = |S|$, the analysis of [8]shows that s is concentrated around $(n/4)^{1/3}$. Furthermore, every labelled cubic graph on m vertices with s specified edges corresponds to exactly the same number of labelled claw-free cubic graphs on n vertices as (G, S). To apply Theorem 1, S can be turned into an edge-rooting of G by assigning arbitrary orientations to the edges in S. It follows that G a.a.s. contains a Hamilton cycle passing through all of the edges of S, and hence that H is a.a.s. hamiltonian.

For unlabelled graphs the same correspondence applies between cubic graphs and claw-free cubic graphs, with the concentration for s shifted to $(2n)^{1/3}$ as indicated in [8]. It was first proved by Bollobás [3], and independently in [9], that unlabelled cubic graphs a.a.s. have only the identity automorphism (see also [14] for a slightly simpler proof), so Theorem 1 applies again to show that an unlabelled claw-free cubic graph is a.a.s. Hamiltonian. ∎

2 Small subgraph conditioning method

We use the basic method in [12] and [13]. This is captured by the main theorem in [10], and was improved by Janson [7] by removing several superfluous conditions and making the conclusions of the method more clear. The following is essentially Janson's version (see also [15, Theorems 4.1 and 4.3 and Corollary 4.2]).

First we need a little notation. Suppose that Y is a non-negative integer random variable defined on a space $\mathcal{G}$ with $\mathbf{E}Y \neq 0$. We define a new model $\mathcal{G}^{(Y)}$ with the same underlying set as $\mathcal{G}$ by weighting the probability of each element G by $Y(G)$. That is, the probability of G in $\mathcal{G}^{(Y)}$ equals the probability in $\mathcal{G}$ multiplied by $Y(G)/\mathbf{E}Y$. Thus the probability of an event A in $\mathcal{G}^{(Y)}$ is $\mathbf{E}_{\mathcal{G}}(Y 1_A)/\mathbf{E}_{\mathcal{G}}Y$, where 1_A is the indicator function of A.

Proposition. *Let $\lambda_i > 0$ and $\delta_i \geq -1$, $i = 1, 2, \ldots$, be constants and suppose that for each n there are random variables X_i, $i = 1, 2, \ldots$, and Y, defined on the same probability space $\mathcal{G}$, such that X_i is non-negative integer valued, Y is non-negative and $\mathbf{E}Y > 0$ (for n sufficiently large). Suppose furthermore that*

(a) X_i, $i = 1, 2, \ldots$ are asymptotically independent Poisson random variables with $\mathbf{E}X_i \to \lambda_i$;

(b)

$$\frac{\mathbf{E}(Y[X_1]_{i_1} \cdots [X_k]_{i_k})}{\mathbf{E}Y} \to \prod_{j=1}^{k} (\lambda_j(1+\delta_j))^{i_j}$$

for every finite sequence $i_1, \ldots, i_k$ of non-negative integers;

(c) $\sum_i \lambda_i \delta_i^2 < \infty$;

(d) $$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \leq \exp\left(\sum_i \lambda_i \delta_i^2\right) + o(1) \qquad \text{as } n \to \infty.$$

Then

(i)

$$\frac{Y}{\mathbf{E}Y} \xrightarrow{d} W = \prod_{i=1}^{\infty} (1+\delta_i)^{Z_i} e^{-\lambda_i \delta_i} \qquad \text{as } n \to \infty,$$

where Z_i are independent Poisson variables with $\mathbf{E}Z_i = \lambda_i$ for $i \geq 1$, and the convergence here and in (a) hold jointly, and as a consequence

(ii)

$$\mathbf{P}(Y > 0) = \exp\left(-\sum_{\delta_i = -1} \lambda_i\right) + o(1)$$

provided $Y = 0$ whenever $X_i > 0$ for any i with $\delta_i = -1$, and

(iii)

$$\overline{\mathcal{G}}^{(Y)} \approx \overline{\mathcal{G}}$$

where $\overline{\mathcal{G}}$ is the probability space obtained from $\mathcal{G}$ by conditioning on the event that $X_i = 0$ for each i such that $\delta_i = -1$.

In our proofs we will not deal directly with random graph spaces. Instead, as usual for random regular graphs, we will work with the model originally developed by Bollobás [2], although almost the same model was used earlier by Bender and Canfield [1]. Let M be a set of dn *points* arranged by groups of d into *cells* $v_1, \ldots, v_n$. A *pairing* P is a partition of M into $dn/2$ pairs.

Now distinguish $j \geq 0$ of the pairs, called *root pairs*, and order the two points in each of these pairs. Let $\Omega_{n,d}^{(j)}$ denote the set of all configurations which result. Thus

$$|\Omega_{n,d}^{(j)}| = \frac{(dn)!2^j}{(dn/2)!2^{dn/2}} \binom{dn/2}{j}. \tag{2.1}$$

We regard $\Omega_{n,d}^{(j)}$ as a probability space by giving each element the same probability.

We associate with P the d-regular pseudograph $\pi(P)$ with vertices $v_1, \ldots, v_n$ and an edge joining v_i to v_j for each pair $\{x, y\}$ with $x \in v_i$ and $y \in v_j$. For later use, define $\pi(\{x, y\}) = v_i v_j$ where $x \in v_i$ and $y \in v_j$. Distinguishing pairs in P is equivalent to choosing root edges in $\pi(P)$, ordering the points in the distinguished pairs in P is equivalent to assigning orientations to the corresponding edges in $\pi(P)$, and a cyclic order on the distinguished pairs in P is equivalent to the cyclic order induced on the corresponding edges in $\pi(P)$.

One feature of the pairing model is that the probability of the event that no loops or multiple edges arise in $\pi(P)$ tends towards a constant strictly less than 1 as $n \to \infty$. We call this event Simple. In addition, each d-regular graph (with j oriented root edges) corresponds under π to precisely $(d!)^n$ different pairings (with j distinguished ordered pairs). Thus,

$$\Gamma_n^{(j)} = \Omega_{n,d}^{(j)}|_{\text{Simple}}. \tag{2.2}$$

We also define $\Omega_{n,d}^{<j>}$ from $\Omega_{n,d}^{(j)}$ by in addition providing the set of distinguished pairs with a prescribed cyclic order. Thus

$$|\Omega_{n,d}^{<j>}| = (j-1)!|\Omega_{n,d}^{(j)}|. \tag{2.3}$$

We regard $\Omega_{n,d}^{<j>}$ as a probability space by giving each element the same probability. As with (2.2), we have

$$\Gamma_n^{<j>} = \Omega_{n,d}^{<j>}|_{\text{Simple}}. \tag{2.4}$$

3 Hamilton cycles through oriented root edges

In this section we prove those results relating to Hamilton cycles passing through oriented root edges, respecting their orientation, but with no cyclic ordering imposed.

Proof of Theorem 1 We will assume $j^2/n \to c$ for some $c > 0$. The result for these values of j obviously implies the result for $j = j(n) \geq 0$, since $\mathbf{P}(\mathcal{H})$ is monotonically decreasing as a function of j.

Let $P \in \Omega_{n,3}^{(j)}$. A subset S of P such that $\pi(S)$ is the set of edges of a Hamilton cycle in $\pi(P)$ is called a Hamilton cycle of the pairing P. Let $Y^{(j)} = Y^{(j)}(P)$ denote the number of Hamilton cycles in P which contain all the root pairs and which can be oriented so that the root pairs are traversed in the direction of their orientations. (Traversal of the Hamilton cycle corresponds to traversal of its image under π.) Also, for $i \geq 1$ let $C_i = C_i(P)$ denote the number of $S \subseteq P$ such that $\pi(S)$ is a cycle of length i in $\pi(P)$. Similarly, for $i \geq 2$ let $A_i = A_i(P)$ denote the number of $S \subseteq P$ such that $\pi(S)$ is a path of length i having the end edges belonging to the root set and anti-oriented, i.e. either both oriented toward the center of the path or both oriented away from the center. Let D_i be defined in the same way except that the ends of the path should be di-oriented, *i.e.* with one toward the center and one away.

In the rest of the proof, we will show that conditions (a) to (d) of the Proposition hold in $\Omega_{n,3}^{(j)}$ with the variable Y denoting $Y^{(j)}$ and with the variables X_i of three types: C_i ($i \geq 1$), A_i ($i \geq 2$), and D_i ($i \geq 2$). Thus $X_1, X_2, X_3, \ldots = C_1, C_2, A_2, D_2, C_3, A_3, D_3, \ldots$. The corresponding values of λ_i are $\bar{\lambda}_i$, $\hat{\lambda}_i$ and $\tilde{\lambda}_i$ where

$$\bar{\lambda}_i = \frac{2^{i-1}}{i}, \qquad \hat{\lambda}_i = \tilde{\lambda}_i = \frac{2^{i-1}c}{3}, \tag{3.1}$$

and the corresponding δ_i are $\bar{\delta}_i$, $\hat{\delta}_i$ and $\tilde{\delta}_i$ where

$$\bar{\delta}_i = \tfrac{1}{2^i}((-1)^i - 1), \quad \hat{\delta}_i = \tfrac{3}{2^i}(-1 - \tfrac{1}{3}(-1)^i), \quad \tilde{\delta}_i = \tfrac{3}{2^i}(1 - \tfrac{1}{3}(-1)^i). \tag{3.2}$$

We begin with (a). Simple calculations using factorial moments as in the book by Bollobás [4, Chapter 2] (see also [15, Section 2.3]) show that (a) holds for the variables C_i, A_i and D_i with the corresponding expectations defined in (3.1). For C_i this is equation (1.3) of [12]. For A_i and D_i we can make use of the asymptotically independent Poisson distributions of short cycles to conclude that for $n \to \infty$ most vertices are not within distance i of any cycle of length at most i. Thus the asymptotic number of paths of length i in a random cubic graph of order n is $n \cdot 3 \cdot 2^{i-1}/2$ for $i \geq 2$, as there are n vertices at which to start, three incident edges at the starting vertex and two new edges available to continue from each internal vertex of the path. This overcounts by a factor of 2 since one end of the path was given a special status as the starting vertex. Now, each path of length $i \geq 2$ can be anti-oriented in two ways and di-oriented in two ways. The probability that any particular path with two ends oriented has both of its ends included in a j-edge rooting with the given orientations is exactly $\binom{3n/2-2}{j-2} 2^{j-2} / \binom{3n/2}{j} 2^j = j(j-1)/4(\frac{3n}{2})(\frac{3n}{2}-1)$ which is asymptotic to $\frac{c}{9n}$ as $n \to \infty$ since $j^2/n \to c > 0$. Thus $\hat{\lambda}_i = \tilde{\lambda}_i$ is the product $3n2^{i-2} \cdot 2 \cdot \frac{c}{9n} = 2^{i-1}c/3$. For (b), we first estimate $\mathbf{E}Y^{(0)}$ and $\mathbf{E}Y^{(j)}$ in general. Note that $Y^{(0)}$ is simply the number of Hamilton cycles in a pairing in $\Omega_{n,3}^{(0)}$. We count pairings with a distinguished Hamilton cycle H, and divide by the total number $|\Omega_{n,3}^{(0)}|$ of pairings, given by (2.1). The pairs for H can be chosen in $\frac{6^n(n)!}{2n}$ ways, where $\frac{(n)!}{2n}$ counts the circular arrangements of the cells and in each cell there are then six ways to choose which points in a given cell go in which pairs. The pairing can then be completed by matching the remaining points in $\frac{(n)!}{(n/2)!2^{n/2}}$ ways. This produces

$$\mathbf{E}Y^{(0)} \sim \frac{\sqrt{\pi}}{\sqrt{2n}} \left(\frac{4}{3}\right)^{n/2}. \tag{3.3}$$

For $j \geq 1$ and each $P \in \Omega_{n,3}^{(0)}$, the number of ways of distinguishing j pairs and assigning them individual orderings compatible with one of the two possible orientations of H is $2\binom{n}{j}$.

This factor is independent of P, H, and any distinguished short cycles. So in view of (2.1), for $j \geq 1$

$$\mathbf{E}Y^{(j)} = \mathbf{E}Y^{(0)} \frac{2\binom{n}{j}}{2^j \binom{3n/2}{j}}. \tag{3.4}$$

For estimating the numerator of (b), short cycles and short paths connecting root edges are distinguished in addition to H. First, we consider just one short cycle or path.

Since $\Omega_{n,3}^{(0)}$ is the standard model for random cubic graphs, $\mathbf{E}(Y^{(0)}C_i)/\mathbf{E}Y^{(0)}$ is precisely the ratio calculated in [12, equation (2.6)] (or [15, Section 4.2]), where it was shown that

$$\frac{\mathbf{E}(Y^{(0)}C_i)}{\mathbf{E}Y^{(0)}} \to (\bar{\lambda}_i(1+\bar{\delta}_i)), \tag{3.5}$$

verifying (b) in this case. Furthermore, since C_i just counts i-cycles, paying no attention to root edges, it follows as with (3.4) that

$$\mathbf{E}(Y^{(j)}C_i) = \mathbf{E}(Y^{(0)}C_i) \frac{2\binom{n}{j}}{2^j \binom{3n/2}{j}}.$$

Since $\bar{\lambda}_i$ and $\bar{\delta}_i$ are independent of j, this together with (3.4) and (3.5) gives

$$\frac{\mathbf{E}(Y^{(j)}C_i)}{\mathbf{E}Y^{(j)}} \to (\bar{\lambda}_i(1+\bar{\delta}_i)), \tag{3.6}$$

as required in this case.

We next calculate $\mathbf{E}(Y^{(j)}A_i)/\mathbf{E}Y^{(j)}$ and $\mathbf{E}(Y^{(j)}D_i)/\mathbf{E}Y^{(j)}$ by recalculating (3.3) and (3.4) and analyzing the changes introduced (as factors) when an appropriate path of length i is distinguished in addition to the Hamilton cycle. We examine first A_i; the calculation for D_i is almost identical (as we shall see). Lay down the Hamilton cycle H of the pairing with an orientation, then $j-2$ root pairs on H, then an oriented path (of pairs) of length i beginning and ending with edges on H which become the last two root pairs to be chosen, divide by 2 to remove the orientation of the path, and finally fill in with a perfect matching of the remaining unpaired points to complete the cubic configuration. There is no difference in the number of ways to choose H. For the root pair choices, the factor is

$$\binom{n}{j-2}\binom{n}{j}^{-1} \sim \frac{j^2}{n^2} \sim \frac{c}{n}.$$

The oriented path of length i must have its first and last pairs in H. It may take pairs not in H, which we call diagonals, but not consecutively in the path or else a vertex of degree 4 would be created. If b is the number of diagonal pairs in the path, then

$$\binom{i-1-b}{b}$$

is the number of ways to choose the location of the diagonals in the path so that no two are consecutive and the end pairs are not diagonals.

We claim that the path can be laid out in this pattern in asymptotically

$$n \cdot 2 \cdot n^b \cdot 2^{b-1}$$

ways. To justify this is we first show that it is an upper bound. There are $n-(j-2)$ pairs in H eligible as the first edge in the path, two ways for the path to leave this pair, at most n choices of where to end each of the b diagonal pairs, and at most two choices as to which way to follow H after each diagonal except the last. The direction in which to follow H after the last diagonal is determined by the orientation required of the final pair in the path. This is forced by the orientation of H, which determines the orientations of the root pairs, and the fact that the path is to be anti-oriented.

A corresponding lower bound is $(n-j+2)\cdot 2\cdot (n-2bi-i-j)^b\cdot 2^{b-1}$. This is because at any stage in laying down the path we can reserve i vertices on each side of the b segments of the path on H to be avoided in choosing the second end of a diagonal pair as well as the at most i vertices of the path already chosen. This reduces the destination choices to $(n-2bi-i)^b$ and ensures that the path does not self-intersect no matter which of the 2^{b-1} sets of choices of direction are made after the diagonals. In addition, for the last diagonal, j extra destinations must be forbidden so that the path does not end with a pair already chosen as a root pair. This establishes the lower bound. Since i is fixed, $b < i/2$, and $j = O(\sqrt{n})$, the lower and upper bounds are asymptotically equal. Note that $b \geq 1$ since the final pair has to be oriented in opposition to the initial edge. At this point we multiply by

$$\frac{1}{2}$$

to remove the orientation of the path.

To complete the pairing after building a path with b diagonals, note there are $(n-2b)!/(\frac{1}{2}n-b)!2^{\frac{1}{2}n-b}$ different perfect matchings of the remaining $n-2b$ unpaired points. This compares with $n!/(n/2)!2^{n/2}$ without the path. Since b is bounded as $n\to\infty$, the ratio is asymptotic to

$$n^{-b}.$$

The net effect for A_i, asymptotically, is the product of the above five factors summed over b, which simplifies to

$$\begin{aligned}
\mathbf{E}(Y^{(j)}A_i)/\mathbf{E}Y^{(j)} &\sim c\sum_{b\geq 1} 2^{b-1}\binom{i-1-b}{b} \\
&= c\left(-\frac{1}{2}+\frac{1}{2}[z^{i-1}](1-z(1+2z))^{-1}\right) \\
&= c\left(-\frac{1}{2}+\frac{1}{2}[z^{i-1}]\left(\frac{2}{3}(1-2z)^{-1}+\frac{1}{3}(1+z)^{-1}\right)\right) \\
&= c\left(-\frac{1}{2}+\frac{1}{3}2^{i-1}+\frac{1}{6}(-1)^{i-1}\right),
\end{aligned}$$

where square brackets denote extraction of coefficients. This is in accordance with (b) with $\hat{\lambda}_i$ and $\hat{\delta}_i$ given by (3.1) and (3.2), for a single anti-oriented path of length $i \geq 2$.

The corresponding argument for D_i is the same as for A_i except for the inclusion of the $b=0$ term. For D_i, $b=0$ automatically gives a di-oriented path of length i, contributing an extra term asymptotic to c to the summation. Hence

$$\mathbf{E}(Y^{(j)}D_i)/\mathbf{E}Y^{(j)} \sim c\left(\frac{1}{2}+\frac{1}{3}2^{i-1}+\frac{1}{6}(-1)^{i-1}\right),$$

in accordance with (b) and $\tilde{\lambda}_i$ and $\tilde{\delta}_i$ given by (3.1) and (3.2), for a single di-oriented path of length $i \geq 2$.

To verify (b) in general for arbitrary numbers of cycles and paths, one observes that the above arguments for cycles and the two types of paths easily generalize for combinations. The effect due to each individual path or cycle is asymptotically independent of the others, which gives (b) in the general case.

Condition (c) follows by computing the following summations:

$$\sum_{i\geq 1}\bar{\lambda}_i\bar{\delta}_i^2 = \log 3, \qquad \sum_{i\geq 2}\hat{\lambda}_i\hat{\delta}_i^2 = c, \qquad \sum_{i\geq 2}\tilde{\lambda}_i\tilde{\delta}_i^2 = \frac{2c}{3}. \tag{3.7}$$

The exponential of the sum of these is $3e^{5c/3}$.

For condition (d), we follow the pattern of computation in [11, Theorem 2.4].

We must compute $\mathbf{E}(Y^{(j)}(Y^{(j)}-1))$. This is done by choosing an ordered pair of oriented Hamilton cycles (H_1, H_2) in a pairing, both containing the same set of j oriented root edges (with all orientations consistent), then completing the pairing, and finally dividing by the total number of pairings.

First choose H_1 in the pairing, which can be done in

$$\frac{6^n(n)!}{n} \tag{3.8}$$

ways (including a factor of 2 for the orientation). We will next choose the intersection $H_1 \cap H_2$, and also where the root edges are. Afterwards, we will fill in the rest of H_2 and, finally, the remainder of the pairing.

The pairs in $H_1 \cap H_2$ create k paths, for some $k \geq 2$, which contain the j root pairs. Since every cell contains three points, these paths all have length at least 1 and the gaps between them on H_1 have length exactly 1. We will count these configurations of paths on H_1 with one of the gaps distinguished, and then multiply by

$$\frac{1}{k}. \tag{3.9}$$

First choose any pair for the distinguished gap, in one of

$$n \tag{3.10}$$

ways. Beginning with the distinguished gap, we now decide on a linear arrangement of paths containing root pairs and lay out these paths and the other $k-1$ gaps between them, following the orientation of H_1. We finish this step by counting the linear arrangements, as follows.

Suppose that exactly $j-q$ paths contain at least one root pair. Write x to mark a pair used in a path, and y to mark a root pair. The possibilities for a nonempty path with no root pairs are counted by the generating function $\sum_{i\geq 1} x^i = \frac{x}{1-x}$. There are exactly $k-j+q$ of these paths. Those with an arbitrary number of root pairs are counted by $\sum_{i\geq 1}(x+xy)^i = \frac{x+xy}{(1-x-xy)}$. There are $j-q$ paths of this type excluding those with no root pairs, so each of these has generating function

$$\frac{x+xy}{1-x-xy} - \frac{x}{1-x} = \left(\frac{x}{1-x}\right)\left(\frac{y/(1-x)}{1-yx/(1-x)}\right).$$

Multiplying by $\binom{k}{j-q}$ to select which of the paths are to contain at least one root edge, we find that the number of linear arrangements of k paths containing j root pairs and $n-k$ pairs in total, with exactly $j-q$ paths having at least one root pair, is

$$\begin{aligned}
&[x^{n-k}y^j]\binom{k}{j-q}\left(\frac{x}{1-x}\right)^k\left(\frac{y/(1-x)}{1-yx/(1-x)}\right)^{j-q}\\
&=[x^{n-k}]\binom{k}{j-q}\left(\frac{x}{1-x}\right)^k\binom{j-1}{q}x^q(1-x)^{-j}\\
&=\binom{k}{j-q}\binom{j-1}{q}\binom{n-k+j-q-1}{k+j-1}.
\end{aligned} \tag{3.11}$$

Having chosen $H_1 \cap H_2$, the second cycle H_2 can be completed in

$$2^{k-j+q}(k-1)! \tag{3.12}$$

ways. Here $(k-1)!$ occurs because the k pairs creating diagonals with H_1 must form a cycle with the k paths in $H_1 \cap H_2$. The power of 2 reflects the choice of which end to enter the $k-j+q$ paths which do not contain root edges.

Completing the pairing by matching up the remaining points gives the factor

$$\frac{(n-2k)!}{(\frac{1}{2}n-k)!2^{\frac{1}{2}n-k}}. \tag{3.13}$$

Multiplying together (3.8 – 3.13) and dividing by the expression in (2.1) for $d=3$ gives

$$\mathbf{E}(Y^{(j)}(Y^{(j)}-1)) = \frac{6^n(n)!(3n/2)!2^{n-2j}}{(3n)!\binom{3n/2}{j}}\sum_{k,q}\Pi_{k,q} \tag{3.14}$$

where

$$\Pi_{k,q} = \binom{n-k+j-q-1}{k+j-1}\frac{2^{2k+q}(k-1)!^2(n-2k)!(j-1)!}{(\frac{1}{2}n-k)!(j-q)!(k-j+q)!(j-1-q)!q!}. \tag{3.15}$$

We first consider the values of k such that

$$c_1 n \leq k \leq \frac{n}{2} \tag{3.16}$$

for some constant $1/3 > c_1 > 0$, so that in particular

$$j = O(\sqrt{k}). \tag{3.17}$$

Later we show that the other values of k do not contribute significantly.

Consider the important values of q. For $k \geq c_1 n$ and $j = O(\sqrt{n})$,

$$\Pi_{k,q+1}/\Pi_{k,q} \sim \frac{2(j-q)(j-q-1)(n-2k)}{(q+1)k(n-k)}.$$

This is $O(1/q)$. So for asymptotics, q can be taken to be essentially bounded, and

$$\sum_{q \geq 0} \Pi_{k,q} \sim \Pi_{k,0} \exp\left(\frac{2j^2(n-2k)}{k(n-k)}\right). \tag{3.18}$$

We use

$$\binom{n-k+j-1}{k+j-1} = \binom{n-k-1}{k-1} \frac{[n-k+j-1]_j}{[k+j-1]_j}$$

where $[x]_j$ denotes the falling factorial. Then putting (3.18) into (3.14) gives

$$\mathbf{E}(Y^{(j)}(Y^{(j)}-1)) \sim \sum_{k=c_1 n}^{n/2} F_1 F_2$$

where

$$F_1 = \frac{6^n n!(3n/2)!2^n(n-k-1)!2^{2k}}{(3n)!k(\frac{1}{2}n-k)!}$$

and

$$F_2 = \frac{[n-k+j-1]_j [k]_j}{[3n/2]_j 2^{2j} [k+j-1]_j} \exp\left(\frac{2j^2(n-2k)}{k(n-k)}\right).$$

From first principles, or else from Stirling's formula with remainder, we have

$$[m]_j = m^j e^{-j^2/2m}(1 + O(j/m)) \tag{3.19}$$

as long as $m \to \infty$ and $j = O(\sqrt{m})$. This can be used to evaluate the factorials in the above expressions, in view of (3.16) and (3.17). Stirling's formula is valid provided the argument tends to infinity, but gives a $O(1)$ bound in any case. As we shall see later, the largest terms in the summation only occur when all arguments do go to infinity, which validates this step. For $k \sim n/3$ (to be justified shortly) we obtain

$$F_1 \sim \frac{9 \cdot 2^{n+k}\sqrt{\pi n}\left(1-\frac{k}{n}\right)^{n-k}}{3^{n/2}n^2\left(1-\frac{2k}{n}\right)^{\frac{1}{2}n-k}}$$

and

$$F_2 \sim \frac{(n-k)^j}{[3n/2]_j 2^{2j}} \exp\left(\frac{j^2}{2(n-k)} - \frac{j^2}{k}\right) \exp\left(\frac{2j^2(n-2k)}{k(n-k)}\right).$$

Thus we have

$$\mathbf{E}(Y^{(j)}(Y^{(j)}-1)) \sim \frac{2^n 9\sqrt{\pi}}{3^{n/2}n^{3/2}[3n/2]_j 2^{2j}} \sum_{k=c_1 n}^{n/2} S_k T_k \tag{3.20}$$

where

$$S_k = \frac{2^k\left(1-\frac{k}{n}\right)^{n-k}(n-k)^j}{\left(1-\frac{2k}{n}\right)^{\frac{1}{2}n-k}}$$

and

$$T_k = \exp\left(\frac{j^2}{2(n-k)} + \frac{j^2}{k}\left(\frac{2(n-2k)}{n-k}-1\right)\right).$$

By standard analysis of this summation, as in [11], we find that the peak is at $k = n/3+O(\sqrt{n})$ and that terms outside this range are not significant. (Note that since $n-k \geq n/2$ and $j^2 = O(n)$ the factor T_k is bounded and does not affect the calculation of the peak.) We obtain

$$\sum_{k=c_1 n}^{n/2} S_k \sim \frac{2^n}{3^{n/2}} \cdot \frac{2^j n^j}{3^j} e^{\frac{j^2}{4n}} \frac{\sqrt{4\pi n}}{3},$$

and also we find that everywhere near the peak,

$$T_k \sim e^{\frac{3j^2}{4n}}.$$

On the other hand, the contribution to the right side of (3.14) from terms in the sum for which $k \sim cn$ can be estimated crudely using Stirling's formula, in view of (3.17), to be

$$\left(\frac{2^{1/2+2c}(1-c)^{1-c}}{3^{1/2}(1/2-c)^{1/2-c}} + o(1)\right)^n$$

and it easily follows that the contribution for $k < c_1 n$ is asymptotically negligible for sufficiently small $c_1 > 0$. Thus by (3.3), (3.4) and (3.20), using (3.19) several times,

$$\frac{\mathbf{E}(Y^{(j)}(Y^{(j)}-1))}{(\mathbf{E}Y^{(j)})^2} \sim 3e^{\frac{5j^2}{3n}} \sim 3e^{\frac{5c}{3}}, \tag{3.21}$$

as required for part (d) in view of the calculation in part (c).

We have now established conditions (a) to (d) of the Proposition with the λ_i and δ_i defined as in (3.1) and (3.2). Hence by the Proposition (ii),

$$\mathbf{P}(Y > 0) = \exp(-\bar{\lambda}_1 - \hat{\lambda}_2) + o(1)$$

since $\bar{\delta}_1$ and $\hat{\delta}_2$ are the only δ_i which equal -1 and if $C_1 > 0$ (there is at least one loop) or $A_2 > 0$ (there is at least one anti-oriented path of length 2) then $Y = 0$ (there is no hamilton cycle containing all of the root edges which has an orientation consistent with all of the root edge orientations). By the joint convergence mentioned in part (i) of the Proposition, conditioning on the event Simple, i.e. $C_1 = C_2 = 0$, changes this probability to $\exp(-\hat{\lambda}_2) + o(1) = \exp(-2c/3) + o(1)$. Theorem 1 follows. ∎

Proof of Theorem 4 Again working in the space $\Omega_{n,3}^{(j)}$, put $X_i = C_i$, $\lambda_i = \bar{\lambda}_i$ and $\delta_i = \bar{\delta}_i$ for $i \geq 1$. Then from the proof of Theorem 1, the hypotheses of the Proposition all hold if $j = o(\sqrt{n})$.

We next modify the argument for the space $\Omega_{n,3}^{(j)\backslash\{m\}}$, which is defined as the uniform probability space of pairings in $\Omega_{n,3}^{(j)}$ with j oriented root pairs, and with m other pairs marked. Take j and m to be $o(\sqrt{n})$. Define $Y^{(j)\backslash\{m\}}$ to be the number of Hamilton cycles in a pairing P in $\Omega_{n,3}^{(j)}$ which, as before, contain the root pairs with compatible orientations, and additionally avoid all marked pairs. Keep $X_i = C_i$, $\lambda_i = \bar{\lambda}_i$ and $\delta_i = \bar{\delta}_i$ as at the start of this proof. Simple calculations show that

$$|\Omega_{n,3}^{(j)\backslash\{m\}}| = |\Omega_{n,3}^{(j)}| \binom{\frac{3}{2}n - j}{m} \sim |\Omega_{n,3}^{(j)}| \frac{(3n)^m}{m!2^m} \tag{3.22}$$

and

$$\mathbf{E}Y^{(j)\backslash\{m\}} \sim \mathbf{E}Y^{(j)}3^{-m}. \tag{3.23}$$

The distribution of X_i is exactly the same as before, so part (a) of the Proposition holds. We find similarly $\mathbf{E}(Y^{(j)\backslash\{m\}}C_i) \sim \mathbf{E}(Y^{(j)}C_i)3^{-m}$, and so, from (3.6) and (3.23),

$$\frac{\mathbf{E}(Y^{(j)\backslash\{m\}}C_i)}{\mathbf{E}Y^{(j)\backslash\{m\}}} \to (\lambda_i(1+\delta_i))$$

as required for part (b) of the Proposition. A similar argument yields the required analogue for higher moments. The calculation for (c) is $\sum_{i\geq 1} \lambda_i \delta_i^2 = \log 3$, as in (3.7).

For part (d), the argument leading to (3.14) is valid for $Y^{(j)\backslash\{m\}}$ but with an extra factor $\binom{\frac{1}{2}n-k}{m}$ multiplying (3.13) for the number of ways of choosing the m marked edges from the $\frac{1}{2}n - k$ edges added at the end to $H_1 \cup H_2$. The effect of this extra factor is similar to that of the factors involving j when $j = o(\sqrt{n})$: it does not shift the peak value of k in (3.20) significantly, but multiplies all terms in the summation near the peak by $(n/6)^m/m!$. So we obtain in place of (3.21), using (3.22) and (3.23) and recalling $j = o(\sqrt{n})$,

$$\frac{\mathbf{E}(Y^{(j)\backslash\{m\}}(Y^{(j)\backslash\{m\}}-1))}{(\mathbf{E}Y^{(j)\backslash\{m\}})^2} \sim 3.$$

Thus, the Proposition can be applied, and by (iii) we deduce that

$$\left(\overline{\Omega_{n,3}^{(j)\backslash\{m\}}}\right)^{(Y^{(j)\backslash\{m\}})} \approx \overline{\Omega_{n,3}^{(j)\backslash\{m\}}}.$$

Since $\delta_i = -1$ only for $i = 1$, the latter is the restriction of $\Omega_{n,3}^{(j)\backslash\{m\}}$ to the event $C_1 = 0$. Since the probability of $C_2 = 0$ tends to a non-zero constant, we can further restrict to this event, and by the definition of contiguity and the equivalence of the Ω and Γ models obtain

$$\left(\Gamma_{n,3}^{(j)\backslash\{m\}}\right)^{(Y^{(j)\backslash\{m\}})} \approx \Gamma_{n,3}^{(j)\backslash\{m\}}.$$

The probability space on the left is exactly the same as $\Gamma_{n,\mathrm{ham}}^{(j)} \oplus \Gamma_{n,1}^{\{m\}}$, since in both cases the probability of a graph occurring is proportional to the number of edge-disjoint decompositions into a Hamilton cycle containing all root edges with consistent orientation, and a matching containing all marked edges. (See [15, Section 4.3] for similar examples.) ∎

The Remark after Theorem 5 as it pertains to Theorem 4 is justified by omitting the final restriction to $X_2 = 0$ in the above proof. This shows that

$$\Omega^{(j)}_{n,\mathrm{ham}} + \Omega^{\{m\}}_{n,1} \approx \Omega^{(j)\backslash\{m\}}_{n,3}|_{C_1=0},$$

where the definitions of these spaces should be obvious. Translating to multigraphs we obtain

$$\Gamma^{(j)}_{n,\mathrm{ham}} + \Gamma^{\{m\}}_{n,1} \approx \pi(\Omega^{(j)\backslash\{m\}}_{n,3}|_{C_1=0})$$

where the elments of the space on the right have the distribution of $\pi(P)$ for $P \in \Omega^{(j)\backslash\{m\}}_{n,3}|_{C_1=0}$. The latter space is not uniform, but Janson [7, Theorem 12] showed that it is contiguous to uniformly distributed random d-regular multigraphs. The same reasoning applies for Theorem 5.

Proof of Theorem 5 For $P \in \Omega^{(j)\backslash\{m\}}_{n,d}$, define $Y^{\backslash(j)\backslash\{m\}}$ to be the number of perfect matchings in P which avoid all root edges and marked edges, and $X_i = C_i$ as in the proof of Theorem 4. The calculations in [13] show that for $j = m = 0$ the Proposition applies, with $\lambda_i = \frac{(d-1)^i}{2i}$ and $\delta_i = \frac{(-1)^i}{(d-1)^i}$. (These are however based in the graph spaces, but conversion to spaces of pairings is easy; see also Janson [7, Theorem 4]). The modifications of this argument to incorporate forbidden root edges and marked edges are just like the proof of Theorem 4, so we sketch the details. Compared to the case when $j = m = 0$, the cardinality of $\Omega^{(j)\backslash\{m\}}_{n,d}$ is multiplied by $2^j\binom{dn/2}{j+m}\binom{j+m}{m}$ and so the expectation is multiplied by $[(d-1)n/2]_{j+m}/[(dn/2]_{j+m} \sim (\frac{d-1}{d})^{j+m}$ since $j + m = o(\sqrt{n})$. That is,

$$\frac{\mathbf{E}Y^{\backslash(j)\backslash\{m\}}}{\mathbf{E}Y^{\backslash(0)\{0\}}} \sim \left(\frac{d-1}{d}\right)^{j+m}.$$

Also $\mathbf{E}(Y^{\backslash(j)\backslash\{m\}}C_i) \sim \mathbf{E}(Y^{\backslash(0)\{0\}}C_i)(\frac{d-1}{d})^{j+m}$. For part (d) of the Proposition, the calculation for the case $j = m = 0$ by Bollobás and McKay [5] is similar to the calculations above for (d) in the proof of Theorem 1. The summation in that case is over the number l of edges in the first matching not in the second. The peak occurs at $l = (d-1)n/(2d)$, where the number of edges in neither matching is asymptotically $\frac{n}{2}(d-2+\frac{1}{d}) = \frac{dn}{2}(d-1)^2/d^2$. From here it is easy to show that

$$\frac{\mathbf{E}(Y^{\backslash(j)\backslash\{m\}}(Y^{\backslash(j)\backslash\{m\}}-1))}{\mathbf{E}(Y^{\backslash(0)\{0\}}(Y^{\backslash(0)\{0\}}-1))} \sim \frac{(d-1)^{2(j+m)}}{d^{2(j+m)}},$$

and hence

$$\frac{\mathbf{E}(Y^{\backslash(j)\backslash\{m\}}(Y^{\backslash(j)\backslash\{m\}}-1))}{(\mathbf{E}Y^{\backslash(j)\backslash\{m\}})^2} \sim \frac{\mathbf{E}(Y^{\backslash(0)\{0\}}(Y^{\backslash(0)\{0\}}-1))}{(\mathbf{E}Y^{\backslash(0)\{0\}})^2}.$$

The rest of the proof is analogous to the proof of Theorem 4. ∎

Proof of Theorem 3(i) From Theorems 4 and 5 and the obvious associativity of $\oplus$, we have for $d \geq 3$, n even and $m + j = o(\sqrt{n})$

$$\Gamma^{(j)\backslash\{m\}}_{n,d} \approx \Gamma^{(j)}_{n,\mathrm{ham}} \oplus \Gamma^{\{m\}}_{n,1} \oplus (d-3)\Gamma^{(0)}_{n,1},$$

where $i\Gamma = \Gamma \oplus \cdots \oplus \Gamma$ (i times). This also uses [15, Lemma 4.14], which implies that in this case substituting a space by a contiguous one when applying $\oplus$ gives a contiguous result. Since the probability of a Hamilton cycle with the required property is trivially 1 in the space $\Gamma^{(j)}_{n,\mathbf{ham}} \oplus \Gamma^{\{m\}}_{n,1}$, the theorem follows by the definition of contiguity for n even.

This leaves the case of n odd and hence d even, for which the following argument suffices, as in [13]. Take any graph $G \in \Gamma^{(j)\backslash\{m\}}_{n,d}$, pick a random vertex v, and randomly associate the edges at v into pairs vu_i and vu'_i ($i = 1, \ldots, d/2$). Deleting v and adding the edges $u_i u'_i$ a.a.s. gives a d-regular graph G', since the expected number of short cycles is bounded. Make the new edge $u_1 u'_1$ a new root edge, and make the others new marked edges. This is now (a.a.s.) an element of $\Gamma^{(j+1)\backslash\{m+d/2-1\}}_{n,d}$. Furthermore, it is easy to see that the graphs in the latter space are generated almost uniformly by this process. So by the result for n even, there is a.a.s. a Hamilton cycle in G' behaving as required with respect to the root edges and the marked edges. This induces the required Hamilton cycle in G, since it uses exactly one of the new edges created in the formation of G' from G. ∎

4 Root edges with a cyclic orientation

Proof of Theorem 2 We begin with a result which permits us to eschew consideration of the effect of short paths joining root edges for $j = o(n^{2/5})$, the range of the number of root edges assumed in the hypothesis of the theorem. This in turn allows us to simplify the analysis, at the point where the terms with $q = 0$ (as in the proof of Theorem 1) suffice to give an asymptotic upper bound on the second moment.

Lemma 1 *If $j = o(n^{2/5})$ root edges are randomly chosen from any set S of at least ϵn edges ($\epsilon > 0$ constant) in any cubic graph with n vertices, the probability that some pair of edges are both contained in a path of length at most $w = \lceil \frac{1}{5} \log n \rceil$ tends to 0 as $n \to \infty$.*

Proof. An upper bound for the number of paths of length $i \geq 2$ in any cubic graph on n vertices is $3n2^{i-2}$ in view of the degree of each vertex being 3. Summing over $2 \leq i \leq w$ gives a total upper bound of $3n2^{w-1} = O(n^{6/5})$. For any particular path, the probability that both end edges are root edges is $O\left((j/\epsilon n)^2\right) = o(n^{-6/5})$. Thus, after a random choice of j root edges, the expected number of such paths with both end edges being root edges is $o(1)$, as required. ∎

Defining $w = \lceil \frac{1}{5} \log n \rceil$ as in Lemma 1, consider the uniform probability space $\widehat{\Omega}^{<j>}_{n,3}$ obtained from $\Omega^{<j>}_{n,3}$ by restricting to those pairings P in which every path containing two root edges of $\pi(P)$ has length at least $w + 1$. By Lemma 1,

$$|\widehat{\Omega}^{<j>}_{n,3}| \sim |\Omega^{<j>}_{n,3}|. \tag{4.1}$$

Define C_i as in the proof of Theorem 1. For $P \in \Omega^{<j>}_{n,3}$, let $Y^{<j>} = Y^{<j>}(P)$ denote the number of Hamilton cycles in P for which the image under π contains all the root edges and which can be oriented so that the root edges are traversed in the prescribed cyclic order, and in the direction of their orientations.

We define $C^{<j>}_i = C^{<j>}_i(P)$ for $P \in \Omega^{<j>}_{n,3}$ exactly the same way as C_i for $P \in \Omega^{(j)}_{n,3}$ in the proof of Theorem 1. Finally, we define $Y^{<j>}$ and $C^{<j>}_i$ on the space $\widehat{\Omega}^{<j>}_{n,3}$ by inheritance from $\Omega_{n,3}^{<j>}$. In the rest of the proof, we will show that conditions (a) to (d) of the Proposition hold in $\widehat{\Omega}_{n,3}^{<j>}$, for $X_i = C_i^{<j>}$, $\lambda_i = \bar{\lambda}_i$ and $\delta = \bar{\delta}_i$ as in (3.1) and (3.2).

For $j \geq 1$ and each $P \in \Omega_{n,3}^{(0)}$, the number of ways of distinguishing j pairs, assigning them a cyclic order, and ordering each individual pair, is

$$\binom{3n/2}{j}(j-1)!2^j,$$

independently of the choice of P. Thus the joint distribution of the $C_i^{<j>}$ is exactly the same in $\Omega_{n,3}^{<j>}$ as in $\Omega_{n,3}^{(0)}$.

Note that by (4.1) the difference between $\Omega_{n,3}^{<j>}$ and $\widehat{\Omega}_{n,3}^{<j>}$ is an event in $\Omega_{n,3}^{<j>}$ whose probability is $o(1)$. Thus condition (a) holds in $\widehat{\Omega}_{n,3}^{<j>}$.

For estimating the numerator of (b), short cycles are distinguished in addition to H. For $j \geq 1$ and each $P \in \Omega_{n,3}^{(0)}$, there are $2\binom{n}{j}$ configurations in $\Omega_{n,3}^{<j>}$, obtained by choosing one of the two possible orientations of H and distinguishing j pairs. This induces a cyclic order of root pairs and individual orderings compatible with the orientation of H. This factor is independent of P, H, and any distinguished short cycles. Thus, comparing with (3.3) and (3.4), and using $\mathbf{E}Y^{(0)} = \mathbf{E}Y^{<0>}$,

$$\mathbf{E}Y^{<j>} \sim \frac{\sqrt{\pi}}{\sqrt{2n}}\left(\frac{4}{3}\right)^{n/2}\frac{2\binom{n}{j}}{\binom{3n/2}{j}(j-1)!2^j}, \tag{4.2}$$

and similarly condition (b) holds in $\Omega_{n,3}^{<j>}$ with $X_i = C_i^{<j>}$ and $Y = Y^{<j>}$ (see 3.6).

So again using Lemma 1 with $\epsilon = 1$, condition (b) holds in $\widehat{\Omega}_{n,3}^{<j>}$.

Condition (c) is implied by $\sum_{i\geq 1}\lambda_i\delta_i^2 = \log 3$ (from (3.7)).

For condition (d), the pattern of computation in the proof of Theorem 1 can be followed, but we diverge from this in a subtle way due to the fact that it is difficult to measure the effect of forbidding short paths between root pairs in all phases of the computation.

In the rest of this argument, all expectations are in $\widehat{\Omega}_{n,3}^{<j>}$. We must compute $\mathbf{E}(Y^{<j>}(Y^{<j>} - 1))$. The basic idea is to calculate as in Theorem 1 but with the short path restriction partially imposed when placing the root pairs on the first Hamilton cycle, but not at all imposed when placing the second Hamilton cycle or completing the pairing. This gives an upper bound which suffices for our purposes and which turns out to be asymptotic to the true value. In the resulting expression we will show that we can take $q = 0$ asymptotically.

We find that in the calculations for the analogue of (3.15), (3.8), (3.9) and (3.10) remain unaltered. For the analogue of (3.11), we consider generating the arrangement of paths by the following process. Start with k empty paths of pairs, separated by k pairs. Choose which $j-q$ paths are to contain root pairs. Put one root pair in each of these, and put one non-root pair into each of the other $k-j+q$ paths (there is only one way to place all these pairs). Then pour an extra q root pairs into the chosen $j-q$ paths with repetitions permitted. Also put $w+1$ non-root pairs between every two root pairs within the same path (again in a unique way). Note that these pairs are required by the distance restriction between root pairs within the same path; we don't impose this condition between root pairs in different paths. Finally, pour all the remaining pairs, $n-k-(j-q)-(k-j+q)-q-q(w+1) = n-2k-q(w+2)$

of them, into the $k+j$ subpaths which the root pairs divide the paths into. Thus we obtain in place of (3.11)

$$\binom{k}{j-q}\binom{j-1}{q}\binom{n-k+j-q(w+2)-1}{k+j-1}.$$

Next, since we are ignoring the short path restrictions from this point onwards, (3.12) must be divided by $(j-q-1)!$ since $j-q$ paths contain root pairs, and so the probability that the second Hamilton cycle encounters these paths in the prescribed cyclic order is just $\frac{1}{(j-q-1)!}$. Finally, (3.13) remains the same.

Note that $\mathbf{E}(Y^{<j>}(Y^{<j>}-1))$ is asymptotically at most the sum over k and q of the product of these modifications of (3.8 – 3.13), divided by $|\widehat{\Omega}_{n,3}^{<j>}|$ as given by (2.3) with $d=3$ and (4.1).

We can now argue that $k=\frac{n}{3}+O(n^{3/5})$ along the same lines as in the proof of Theorem 1. Hence, the ratio of successive terms as q is increased by 1 is asymptotically at most

$$\frac{2j^3}{k}\left(\frac{n-2k}{n-k+j-1}\right)^{w+2} \sim \frac{3j^3}{n2^{w+1}} = o(1)$$

as $w \geq \log_2 n^{\frac{1}{5}}$.

Hence, we can take $q=0$ for our asymptotic upper bound, giving (3.15) divided by $(j-1)!^2$. Since $\exp j^2/n$ goes to 1, we are done on comparing (4.2) with (3.3) and (3.4). ∎

Proof of Theorem 3(ii) Based on the proof of Theorem 2, the verification proceeds almost exactly as for Theorem 3(i). ∎

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