1-factorisations of Random Regular Graphs

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Abstract

It is shown that for each $r \geq 3$, a random r-regular graph on $2n$ vertices is equivalent in a certain sense to a set of r randomly chosen disjoint perfect matchings of the $2n$ vertices, as $n \to \infty$. This equivalence of two sequences of probabilistic spaces, called contiguity, occurs when all events almost sure in one sequence of spaces are almost sure in the other, and vice versa. The corresponding statement is also shown for bipartite graphs, and from this it is shown that a random r-regular simple digraph is almost surely strongly r-connected for all $r \geq 2$.

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1 Introduction

Turn the set of labelled r-regular graphs on $2n$ vertices into a probability space $\Omega_{2n,r}$ with the uniform distribution. It was recently shown by the second two authors [12], [13] that for $r \geq 3$ the probability of a random graph in $\Omega_{2n,r}$ being hamiltonian tends to 1 as $n \to \infty$. The proof given there has a simple corollary in terms of a type of asymptotic equivalence of two sequences of spaces called contiguity (defined below). In this paper we distil the essence of the proof technique in [12], [13] as it relates to this asymptotic equivalence, and apply it to a related situation. As a result we conclude a quantitive similarity between a random graph in $\Omega_{2n,r}$ and a random regular graph in the space $\Omega^{+}_{2n,r}$ generated by randomly choosing r disjoint perfect matchings. A similar result is also obtained for bipartite graphs.

Given two sequences $< \Lambda_n >$ and $< \Lambda'_n >$ of probability spaces where Λ_n and Λ'_n have the same underlying sets, Λ_n and Λ'_n are said to be *contiguous* when every event is almost sure in Λ_n iff it is almost sure in Λ'_n (see for example, [8] or [14]). Here "almost sure" refers to $n \to \infty$. The proof of Theorem 2 in [13] immediately implies that for $r \geq 3$, Ω_{2n} is contiguous to the probability space with the same domain as Ω_{2n}, in which each graph occurs with probability proportional to the number of ways in which its edge set can be partitioned into the edges of a k-regular graph and $r - k$ 1-factors, for any $2 \leq k \leq r - 1$. (Actually, only the cases $r \geq 4$ and $k = r - 1$ were treated there explicitly, but the argument covers the case $r = 3 = k + 1$ implicitly, and this case is also treated independently in [10] from the point of view of 2-factors. All the other cases then follow by an easy observation using transitivity of the contiguity relation.) It is natural to ask whether $\Omega_{2n,r}$ is also contiguous to the probability space $\Omega^{+}_{2n,r}$ with again the same domain, in which each graph occurs with probability proportional to the number of ways that its edge set can be partitioned into the edges of r ordered 1-factors. In this paper we show that this is true for $r \geq 3$. Janson [7] has independently given a development of contiguity in this combinatorial context, and an examination of the method introduced in [12] and [13]. Amongst other things, he has obtained the main results of Sections 2 and 3 in the present paper.

The first two authors have independently shown that $\Omega_{2n,r}$ is contiguous to the probability space with the same domain, in which each graph occurs with probability proportional to the number of ways in which its edge set can be partitioned into the edges of a 2-factor, and an $(r - 2)-$factor. (See [10].) This, along with the results of [13] and this paper, implies that for every partition $r_1 + r_2 + \ldots + r_k = r \geq 3$, $\Omega_{2n,r}$ is contiguous to the probability space with the same domain where each member occurs with probability proportional to the number of ways that its edge set can be partitioned into the edges of an r_1-factor, an r_2-factor, ..., and an r_k-factor (where the factors may be taken as ordered).

In the remainder of this introduction we give a theorem which abstracts the main outlines of the approaches taken in [12] and [13]. Then in Section 2 we take the main step in the proof of our new result for graphs, showing that $\Omega_{2n,r}$ and $\Omega^{+}_{2n,r}$ are contiguous for $r = 3$. The case of $r > 3$ will then follow from the results in [13] (Section 3). Analogous results for bipartite graphs are given in Section 4.

Throughout this paper, asymptotics are for $n \to \infty$, unless otherwise specified.

Let $Y, X_1, X_2, \ldots$ be non-negative integer random variables defined on a sequence $\{\Lambda_n\}$ of probability spaces. The following theorem applies analysis of variance to Y as conditioned by specified values of $X_1, X_2, \ldots, X_k$ in order to derive sufficient conditions for Y's value to be asymptotically almost surely close (in ratio) to its conditional expectation. For the applications in [12], [13] and the present paper, Λ_n is a space of r-regular graphs on $2n$ nodes and X_i is the number of $(i+2)$-cycles in a graph. In these same applications, Y is respectively the number of hamilton cycles, the number of perfect matchings, and the number of ordered disjoint pairs of perfect matchings.

The expectation operator for Λ_n is denoted $\mathbf{E}_n$, or more usually just as $\mathbf{E}$ since n is only significant asymptotically. Similarly, $\mathbf{P}_n$ or $\mathbf{P}$ denotes probability over Λ_n. In what follows, $[n]_r$ denotes the falling factorial

$$n(n-1)\cdots(n-r+1).$$

There are two sequences of constants, $\{\lambda_i\}_{i=1,2,\ldots}$and $\{\mu_i\}_{i=1,2,\ldots}$, which will depend on the particular application.

The hypotheses for Theorem 1 are:

$$\mathbf{E}_n Y > 0 \text{ for all } n; \tag{1.1}$$

$$\mathbf{E} X_i \to \lambda_i \text{ for } i \geq 1; \tag{1.2}$$

for any fixed $k \geq 1$

$$X_1, X_2, \ldots, X_k \text{ are asymptotically independent Poisson variables;} \tag{1.3}$$

$$\frac{\mathbf{E}(Y[X_1]_{i_1} \cdots [X_k]_{i_k})}{\mathbf{E}Y} \to \prod_{j=1}^{k} (\lambda_j + \mu_j)^{i_j} \tag{1.4}$$

for any $k \geq 1$ and $i_1, \ldots, i_k \geq 0$;

$$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \to \exp\left(\sum_{i=1}^{\infty} \frac{\mu_i^2}{\lambda_i}\right) < \infty; \tag{1.5}$$

$$\lambda_i > \max\{0, -\mu_i\} \text{ for all } i \geq 1; \tag{1.6}$$

$$\sum_{i=1}^{\infty} \exp\left(-\alpha \lambda_i^{1/3}\right) < \infty \text{ for any } \alpha > 0; \tag{1.7}$$

$$\frac{1}{k} \sum_{i=1}^{k} |\mu_i| \to 0 \text{ as } k \to \infty; \tag{1.8}$$

$$\sum_{i=1}^{\infty} |\mu_i| \lambda_i^{-1/3} < \infty; \tag{1.9}$$

$$\frac{\sum_{i=1}^{k} |\mu_i|}{\ln \sum_{i>k} \mu_i^2/\lambda_i} \to 0 \text{ as } k \to 0. \tag{1.10}$$

Theorem 1. *Given (1.1) - (1.10),*

$$\lim_{\epsilon \to +0} \left\{ \lim_{n \to \infty} \mathbf{P}_n \left(\epsilon < \frac{Y}{\mathbf{E}Y} < \frac{1}{\epsilon} \right) \right\} = 1.$$

In particular, $\mathbf{P}_n(Y > 0) \to 1$ *as* $n \to \infty$.

Theorem 1 yields the following useful corollary.

Corollary 1 *Given (1.1) - (1.10),* Λ_n *is contiguous with the probability space with the same domain as* Λ_n*, in which each element* w *occurs with probability proportional to* $Y(w)$.

The proof of this corollary will follow the proof of Theorem 1. We provide one application of this corollary in Section 5 of the present paper. For other applications, see [5], [9] and [6].

Proof of Theorem 1. We use the techniques introduced in [12] and [13].

For any k, and any k-tuple of positive integers $a = (a_1, \ldots, a_k)$, we define $\mathcal{T}_a$ to be the event $X_i = a_i$, $i = 1, \ldots, k$, and we set $\pi_a = \mathbf{P}(\mathcal{T}_a)$. We also define the conditional mean and variance:

$$\begin{aligned} E_a &= \mathbf{E}(Y|\mathcal{T}_a) \\ V_a &= \mathbf{V}(Y|\mathcal{T}_a). \end{aligned}$$

Note that

$$\mathbf{V}Y = \sum_{a \in N^k} \pi_a V_a + \sum_{a \in N^k} \pi_a E_a^2 - (\mathbf{E}Y)^2 \tag{1.11}$$

We will concentrate our attention on k-tuples for which each a_i is not too much bigger than λ_i, so we define

$$S(y,k) = \{a \mid 0 \le a_i \le \lambda_i + y\lambda_i^{2/3} \text{ for } i = 1, ..., k\}.$$

Fixing y, k, we set $\overline{\mathcal{T}} = \cup_{a \notin S(y,k)} \mathcal{T}_a$, and $\overline{\pi} = \mathbf{P}(\overline{\mathcal{T}})$.

The essence of our proof lies in the following two lemmas. Lemma 1 shows that (a) $\overline{\pi}$ is small, and (b) if $a \in S(k,y)$ then E_a is large. Lemma 2 shows that most of the variance can be explained by the variance between groups, i.e. for each $a \in S(y,k)$, V_a is small, and so conditional on $\mathcal{T}_a$, Y is concentrated around E_a. The proof then follows by taking $y, k \to \infty$.

Set $\chi(k) = \sum_{i=1}^{b} |\mu_i|$, $\varphi(k) = \sum_{i=k+1}^{\infty} \mu_i^2/\lambda_i$, and $\psi(k) = \max\{\exp(-k), \varphi(k)\}$. Then conditions (1.8) and (1.10) are equivalent to $\chi(k) = o(k)$ and $\chi(k) = o(\ln \varphi(k))$, as $k \to \infty$.

Lemma 1 *If y is sufficiently large then*

(a) $\overline{\pi} \le e^{-\alpha y}$ for some absolute constant $\alpha > 0$, and

(b) $a \in S(y,k)$ implies $E_a \ge e^{-(\gamma y + \chi(k))}\mathbf{E}Y$, for some absolute constant $\gamma > 0$.

Lemma 2 *If y is sufficiently large then*

$$\sum_{a \in S(y,k)} \pi_a V_a^2 \le (\delta\psi(k) + o(1))(\mathbf{E}Y)^2,$$

for some absolute constant $\delta > 0$, where the $o(1)$ term is taken as $n \to \infty$.

Upon proving these two lemmas, the rest follows from an application of Chebychev's inequality. Define the random variable $\hat{Y}$ by

$$\hat{Y} = E_a \text{ if } \mathcal{T}_a \text{ holds.}$$

For any $t > 0$,

$$\begin{aligned}
\mathbf{P}(|Y - \hat{Y}| \ge t) &\le \mathbf{E}((Y-\hat{Y})^2)/t^2 \\
&= \sum_{a \in N^k} \pi_a V_a / t^2 \\
&\le (\delta\psi(k) + o(1))(\mathbf{E}Y)^2/t^2,
\end{aligned}$$

by Lemma 2. Now, setting $t = \frac{1}{2}e^{-(\gamma y + \chi(k))}\exp Y$, where γ is from Lemma 1, we obtain

$$
\begin{aligned}
\mathbf{P}(Y < t) &\leq \overline{\pi} + \mathbf{P}(|Y - \hat{Y}| \geq t) \\
&\leq e^{-\alpha y} + 4(\delta\psi(k) + o(1))e^{-2(\gamma y + \chi(k))}.
\end{aligned}
$$

Since $\exp(\chi(k)) = o(\ln(\psi(k)))$ as $k \to \infty$, we can choose $k(y)$ large enough that

$$
4(\delta\psi(k) + o(1))e^{-2(\gamma y + \chi(k))} \leq e^{-c_0 y}.
$$

Thus, for any $\epsilon, \epsilon' > 0$, we can choose y, k large enough that

$$
\lim_{n\to\infty} \mathbf{P}(Y < \epsilon \mathbf{E} Y) < \epsilon',
$$

and so

$$
\lim_{\epsilon\to+0} \left\{ \lim_{n\to\infty} \mathbf{P}_n \left(\epsilon < \frac{Y}{\mathbf{E}Y} \right) \right\} = 1.
$$

The other part of Theorem 1 follows from a straightforward application of Markov's inequality.

It only remains to prove Lemmas 1 and 2. First, we need a somewhat more general lemma to replace Lemma 3 of [12, 13].

Lemma 3 *Let $\eta_1, \eta_2, \ldots > 0$ be given such that for any $\alpha > 0$*

$$
\sum_{i=1}^{\infty} \exp(-\alpha \eta_i^{1/3}) < \infty.
$$

Then uniformly over $x \geq 1$,

$$
R(x) = \sum_{i=1}^{\infty} \sum_{t > \eta_i + y_i} \eta_i^t / (t! \exp(\eta_i)) = O(\exp(-c_0 x))
$$

where $y_i = x\eta_i^{2/3}$ and $c_0 > 0$ is independent of x.

Proof. If we simply follow the proof of Lemma 3 in [12], the result above is obtained with the constant

$$
c_0 = \frac{1}{4} \min\{\eta_i^{1/3}, \eta_i^{2/3} : i \geq 1\}.
$$

The bounded summation hypothesis implies $\eta_i \to \infty$ as $i \to 0$. With $\eta_i > 0$ for all $i \geq 1$, this implies that the minimum in the definition of c_0 exists and is strictly positive. This proves the lemma. ∎

Proof of Lemma 1

(a) This is our analog of (2.1) of [12] (see also (2.4) of [13]). As in [12], it follows immediately from Lemma 3 and hypothesis (1.3) of Theorem 1.

(b) We first state an analog of Lemma 2 of [12] and [13] (the proof being identical to the proof found there, after applying hypotheses (1.4) and (1.6) of Theorem 1, where (1.4) is our analog of (2.8) in [12], and (1.6) ensures $\lambda_j + \mu_j > 0$ for each j):

$$E_a \sim \mathbf{E}Y \prod_{i=1}^{k} \left(1 + \frac{\mu_i}{\lambda_i}\right)^{a_i} e^{-\mu_i}. \tag{1.12}$$

Now, it is sufficient to bound $\prod_{i=1}^{k}\left(1+\frac{\mu_i}{\lambda_i}\right)^{a_i} e^{-\mu_i}$ appropriately, for all $a \in S(y,k)$. Note that

$$\prod_{i=1}^{k} \left(1 + \frac{\mu_i}{\lambda_i}\right)^{a_i} e^{-\mu_i} \geq ABC^y$$

where

$$\begin{aligned} A &= \prod_{\mu_i<0} \left(1 + \frac{\mu_i}{\lambda_i}\right)^{a_i} e^{-\mu_i}, \\ B &= e^{-\sum_{\mu_i>0} \mu_i}, \\ C &= \prod_{\mu_i<0} \left(1 + \frac{\mu_i}{\lambda_i}\right)^{\lambda_i^{2/3}}. \end{aligned}$$

For any y, k, $A \geq \exp\left(-\sum_{i=1}^{\infty} \frac{\mu_i^2}{2\lambda_i}\right)$ which converges to a constant greater than 1 by hypothesis (1.5) of Theorem 1, and $B \geq \exp(-\chi(k))$. Also,

$$C \geq \exp\left(-\sum_{i=1}^{\infty} |\mu_i| \lambda_i^{-1/3} - \frac{1}{2}\mu_i^2 \lambda_i^{-4/3}\right) > \exp\left(-\frac{3}{2}\sum_{i=1}^{\infty} |\mu_i| \lambda_i^{-1/3}\right)$$

which converges to a constant greater than 0 by hypothesis (1.9) of Theorem 1, thus establishing Lemma 1(b). ∎

Proof of Lemma 2

The following is our analog of (2.4) of [12] (see also (2.7) of [13]). The proof is the same the proof found there, after applying (1.12), and Lemma 3:

$$\mathbf{E}(E_{(X_1,\ldots,X_k)})^2 \geq (\mathbf{E}Y)^2 \left(\exp\left(\sum_{i=1}^{\infty} \frac{\mu_i^2}{\lambda_i}\right)\right)(1 - O(\psi(k)) + o(1)), \tag{1.13}$$

where the constant implicit in $O(\psi(k))$ is independent of k. Applying (1.11) and hypothesis (1.5) of Theorem 1, we immediately obtain

$$\mathbf{E}V_{(X_1,\ldots,X_k)} = (\mathbf{E}Y)^2(O(\psi(k)) + o(1)),$$

thus yielding Lemma 2. ∎

Proof of Corollary 1. Denote the domain of Λ_n by Ω and denote by Λ_n^+ the probability space with domain Ω in which each element w occurs with probability proportional to

$Y(w)$; i.e., with probability $Y(w)/(\mathbf{E}Y|\Omega|)$. In what follows, we use $\mathbf{P}$ and $\mathbf{P}^+$ to denote probabilities taken in Λ_n, and Λ_n^+ respectively.

Suppose that a property P occurs with probability $\rho(n) = o(1)$ in Λ_n. From hypothesis (1.5), we have

$$\mathbf{V}Y = \zeta = \exp\left(\left(\sum_{i=1}^{\infty}\frac{\mu_i^2}{\lambda_i} + o(1)\right) - 1\right)(\mathbf{E}Y)^2 = O((\mathbf{E}Y)^2).$$

Therefore, by Chebychev's inequality, for each positive integer i, the number of elements $w \in \Omega$ for which $Y(w) > i\mathbf{E}Y$ is at most $\frac{\zeta}{i^2}|\Omega|$. Thus, taking $I = \{i \mid \frac{\zeta}{i+1^2}|\Omega| < \rho(n)\}$, we have

$$\begin{aligned} \mathbf{P}^+(P) &\leq \rho(n)\sqrt{\frac{\zeta}{\rho(n)}} + \sum_{i\in I}\left(\frac{\zeta}{i-1^2} - \frac{\zeta}{i^2}\right) i \\ &= O\left(\sqrt{\rho(n)}\right) + \sum_{i\in I} O(i^{-2}) \\ &= o(1), \end{aligned}$$

thus establishing one direction of the contiguity.

For the other direction, suppose that a property P holds with probability $\rho(n) = o(1)$ in Λ_n^+.

Denote by Ω^* the set $\{w \in \Omega \mid \rho(n)^{1/3} < Y(w)/\mathbf{E}Y < \rho(n)^{-1/3}\}$. By Chebychev's inequality, for sufficiently large n, $\mathbf{P}^+(\Omega^*) = 1 - o(1)$, and so $\mathbf{P}^+(P|\Omega^*) = \rho(n)(1+o(1))$.

Furthermore, by our bounds on Y, $\mathbf{P}^+(P|\Omega^*) \geq \rho(n)^{2/3}\mathbf{P}(P|\Omega^*)$. Therefore, $\mathbf{P}(P|\Omega^*) \leq \rho(n)^{1/3}(1 + o(1)) = o(1)$, thus establishing the other direction of the contiguity. ∎

2 Triple 1-factors in cubic graphs

A *double 1-factor* is an ordered pair of disjoint 1-factors. In this section we make computations concerning the number of double 1-factors in a random graph in $\Omega_{2n,3}$. (The expectation and variance of the number of 1-factors in $\Omega_{2n,r}$ was found by Bollobás and McKay [3].) A double 1-factor in a cubic graph determines an ordered triple of disjoint 1-factors, and the converse is also true. Thus, from Theorem 2 below and the Corollary to Theorem 1, we conclude the contiguity of $\Omega_{2n,3}$ and $\Omega_{2n,3}^+$.

The number M of labelled cubic graphs on $2n$ vertices is asymptotically

$$M \sim \frac{(6n)!}{e^2(3n)!3^{2n}2^{5n}} \sim \frac{n^{3n}3^n2^n\sqrt{2}}{e^{3n+2}}. \tag{2.1}$$

The calculations here resemble those in [11] and [12]. In the following theorem we take $0^0 = 1$.

Theorem 2. *Let $Y = Y_n$ denote the number of double 1-factors in a random cubic graph on $2n$ vertices, and let X_i denote the number of cycles of length i ($i \geq 3$). Then*

$$\mathbf{E}Y \sim \frac{2^{2n+1}e^{1/2}}{3^n},$$

$$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \to \frac{4}{e^{5/4}}.$$

Furthermore, if we define $\lambda_i = \frac{2^{i-1}}{i}$ and $\mu_i = \frac{(-1)^i}{i}$ for $i \geq 3$, then the hypotheses (1.1) – (1.10) of Theorem 1 all hold with the indices in X_i, λ_i and μ_i all shifted down by 2.

Proof. The number of perfect matchings of the vertices $\{1, 2, \ldots, 2n\}$ is $(2n)!/(n!2^n)$. By the main theorem in Bender and Canfield [2], the number of ways to choose a second perfect matching, avoiding the edges in the first, is asymptotic to $(2n)!/(n!2^n\sqrt{e})$. This gives the asymptotic number of double 1-factors. The number of ways to complete a double 1-factor to a cubic graph is again given by the result of [2] to be asymptotic to $(2n)!/(n!2^n e)$. We obtain the claimed value of $\mathbf{E}Y$ by taking the product of these numbers and dividing by M.

It will next be shown that

$$\mathbf{E}Y^2 \sim \frac{2^{4n+4}}{3^{2n}e^{1/4}}, \tag{2.2}$$

from which the second statement in the theorem follows. We do this by showing that total number T of ordered pairs of distinct double 1-factors in all cubic graphs on $2n$ vertices satisfies

$$T \sim \frac{n^{3n}2^{5n+9/2}}{3^n e^{3n+9/4}}. \tag{2.3}$$

This is computed by counting cubic graphs once for every ordered pair (D_1, D_2) of distinct double 1-factors they contain. Then (2.2) follows by dividing by M and using (2.1).

In [11], the last two authors made an analogous calculation by considering a Hamilton cycle, placing another one in the same graph, and then completing this to a cubic graph G. For the present problem we need to alter this scheme.

Let H_j be the union of the two 1-factors in a double 1-factor D_j, and let A_j and B_j denote the edge sets of these two 1-factors. Thus H_j is a subgraph of G consisting of a set of disjoint even cycles, and A_j and B_j alternate along these cycles. Note that as G is 3-regular, each vertex is incident with either 1 or 2 edges of $H_1 \cap H|2$. Thus, each vertex has degree 1 or 2 in $G - (H_1 \cap H_2)$ and $H_j - (H_1 \cap H_2)$ is a matching, $j = 1, 2$.

We first place the intersection of H_1 and H_2 with the edges in the cycles of $H_1 \cap H_2$ assigned to either A_1 or B_1 and also to A_2 or B_2. Then we similarly assign the edges in the paths of $H_1 \cap H_2$, and then complete the choice of H_1 and H_2. This determines D_1 and D_2, so the final step is to add a matching to complete the choice of G.

Since all cycles in H_j are even, $H_1 \cap H_2$ consists of a set of disjoint paths and even cycles. As the total number of vertices is even, we can let $2i$ denote the number of even-length paths (i.e. those with an odd number of vertices), and we suppose there are k paths in total. The edges of any cycle can be assigned in an alternating manner to A_1 or B_1 and to A_2 or B_2 in four different ways. The exponential generating function counting the number of different possibilities for $H_1 \cap H_2$ is thus

$$\frac{(\frac{1}{2}x^2(1-x^2)^{-1})^{k-2i}}{(k-2i)!} \times \frac{(\frac{1}{2}x^3(1-x^2)^{-1})^{2i}}{(2i)!} \times \frac{e^{-x^2}}{1-x^2},$$

where the first term counts the number of choices for $k-2i$ odd paths, the second term counts the number of choices for $2i$ even paths, and the final term counts the number of choices for a set of even cycles, along with an assignment of their edges to A_1, B_1 and to A_2, B_2. Hence the number of $H_1 \cap H_2$ is

$$\frac{(2n)!}{2^k(k-2i)!(2i)!} \sum_{s \geq 0} \binom{n-i-s}{k} \frac{(-1)^s}{s!}. \tag{2.4}$$

Note here that s indexes the number of cycles in $H_1 \cap H_2$.

For $j = 1$ and 2, it must be decided which of the remaining edges of $H_1 \cap H_2$ are in A_j and which are in B_j. These must alternate along the paths. We denote by $\mathcal{A}_1$ the set of endpoints of the paths in $H_1 \cap H_2$ for which the path-edges they are incident with lie in A_1. We define $\mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2$ in a similar manner. Since the completion of $H_1 \cap H_2$ to H_j consists of a matching, $|\mathcal{A}_j|$ is even, and we denote $|\mathcal{A}_j| = 2a_j$.

For each $j = 1, 2$, given a_j, the number of ways to choose which edges in odd-length paths of $H_1 \cap H_2$ are in A_j is

$$\binom{k-2i}{a_j - i}, \tag{2.5}$$

and for the even-length paths it is

$$2^{2i}. \tag{2.6}$$

Completing $H_1 \cap H_2$ to H_1 consists of adding a matching to $\mathcal{A}_1$, and another matching to $\mathcal{B}_1$. Let d denote the number of paths of $H_1 \cap H_2$ of length 1. These represent the only edges which are forbidden in these matchings. So again using [2], the number of completions to H_1 is equal to

$$\frac{(2a_1)!(2k-2a_1)!}{a_1! 2^{a_1} (k-a_1)! 2^{k-a_1} e^{\kappa_1}} \tag{2.7}$$

where $\kappa_1 \to \frac{1}{2}$ provided $a_1 \sim d \sim k/2 \to \infty$ (a condition which we will later see applies in all relevant cases).

For the completion of H_2, we need to forbid some edges: the d edges as for H_1; and also some of the edges of H_1 added in the completion of H_1, namely those which join two vertices both in $\mathcal{A}_2$ or both in $\mathcal{B}_2$. Let d' denote the number of these edges. Then the number of completions to H_2 is equal to

$$\frac{(2a_2)!(2k-2a_2)!}{a_2!2^{a_2}(k-a_2)!2^{k-a_2}e^{\kappa_2}} \tag{2.8}$$

where $\kappa_2 \to 1$ provided $a_2 \sim d \sim d' \sim k/2 \to \infty$, and $\kappa_2 \geq 0$ always.

Finally to complete $H_1 \cup H_2$ to the graph G we must add a matching of the remaining $2n-2k$ vertices of degree 2. The edges forbidden this time are the "internal" edges of the paths and cycles in $H_1 \cap H_2$, and are $2n-3k+d$ in number. Hence the number of completions is equal to

$$\frac{(2n-2k)!}{(n-k)!2^{n-k}e^{\kappa_3}} \tag{2.9}$$

where $\kappa_3 \to \frac{1}{2}$ provided $d \sim k/2$ and $n-k \to \infty$, and $\kappa_3 \geq 0$ always.

Next, combine (2.4) to (2.9), using Stirling's formula for factorials, and assuming the argument of each factorial, except for s, tends to infinity. This supposition is justified later, when we analyze the values for which F achieves its maximum. We find that the contribution to T from any fixed s in (2.4) is asymptotic to

$$\frac{n^{2n}2^{3n}}{e^{3n}\pi^2}e^{-\kappa_1-\kappa_2-\kappa_3}\sqrt{\alpha}\sum_{k,i,a_1,a_2,d,d'}\left(\frac{-(n-k-i)}{n-i}\right)^s\frac{F}{s!} \tag{2.10}$$

where $F = F(n,k,i,a_1,a_2)$ is defined by

$$F = \frac{2^{2i}g(k-2i)g(a_1)g(k-a_1)g(a_2)g(k-a_2)g(n-i)g(n-k)}{g(a_1-i)g(k-a_1-i)g(a_2-i)g(k-a_2-i)g(n-i-k)g(k)g^2(i)},$$

$g(x) = x^x$, and

$$\alpha = \frac{2n(k-2i)(n-i)}{ki(a_1-i)(a_2-i)(k-a_1-i)(k-a_2-i)(n-i-k)}$$

Simple analysis of F reveals that it achieves its maximum for given n at $k = 2n/3$, $i = n/9$, and $a_1 = a_2 = n/3$. For example, one can apply an analysis similar to that applied to the similar funciton in [11], or one can observe that for fixed i, k, F achieves its maximum at $a_1 = a_2 = k/2$, and after making this substitution, proceed with a straightforward analysis of the derivatives of F with respect to i and k. Expanding about this maximum gives

$$F(n,k,i,a_1,a_2) \sim \left(\frac{4n}{3}\right)^n \exp\left(-\frac{15}{8}\delta_k^2 - \frac{243}{16}\delta_i^2 - \frac{3}{2}\delta_{a_1}^2 - \frac{3}{2}\delta_{a_2}^2 + \frac{3}{2}\delta_{a_1}\delta_k + \frac{3}{2}\delta_{a_2}\delta_k\right)$$

where

$$\delta_k = \frac{k-2n/3}{\sqrt{n}}, \quad \delta_i = \frac{i-n/9}{\sqrt{n}}, \quad \delta_{a_j} = \frac{a_j - n/3}{\sqrt{n}}.$$

The contribution from the neglected terms in this expansion is easily shown to be negligible, and so the sum of F over all relevant k, i, a_1 and a_2 is asymptotic to

$$\left(\frac{4n}{3}\right)^n \frac{\pi^2 n^2 2^{9/2}}{3^{9/2}},$$

and we can take $\alpha = 3^9$.

In addition, for the κ_j we need to know the behaviour of d. The argument about d in [11] applies precisely to the present situation and allows us to conclude that $d \sim n/3$ in almost all of the configurations in which k and i behave asymptotically as determined above. Hence we can assume $\kappa_1 \to \frac{1}{2}$ and $\kappa_3 \to \frac{1}{2}$. To determine the usual value of d', we consider a random valid assignment of edges of $H_1 \cap H_2$ to A_2 and B_2 conditioned on the completion of H_1 to D_1. The conditional probability that any given edge in this completion contributes to d' is exactly $1/2$. A simple limit argument implies that $d' \sim k/2$. Thus $\kappa_2 \to 1$, and so (2.5) is asymptotic to

$$\frac{n^{3n} 2^{5n+9/2}}{3^n e^{3n+2} (-4)^s s!}.$$

It can be seen from (2.4) that the terms with $s \to \infty$ can be neglected. Summing over $s \geq 0$ gives (2.3). This completes the proof of the second statement in the theorem.

The hypothesis (1.3) of Theorem 1 that the X_i are asymptotically independent Poisson variables with means $\lambda_i = 2^{i-1}/i$ (given by Theorem 2) is well known (see [12] for references).

Finally we prove

$$\frac{\mathbf{E}(Y[X_3]_{i_3} \cdots [X_k]_{i_k})}{\mathbf{E}Y} \to \prod_{j=3}^{k} (\lambda_j + \mu_j)^{i_j}. \tag{2.11}$$

This argument follows the derivation of (2.8) in [12]. However, here the even cycles have a different effect.

We first show that

$$\mathbf{E}(YX_m) \sim (\lambda_m + \mu_m)\mathbf{E}Y, \tag{2.12}$$

($m \geq 3$) by counting cubic graphs G with a given double 1-factor D once for every m-cycle C that they contain, and dividing by M. It is clear that there must be s edges of C which are not contained in D for some s in the range $0 \leq s \leq m/2$, since these edges must be mutually non-adjacent.

For the contribution from cycles C with $s > 0$, the proof of Lemma 2 in [12] applies with no change, and gives

$$\frac{2^{m-1}-1}{m}\mathbf{E}Y(1+o(1))$$

for m odd and

$$\frac{2^{m-1}}{m}\mathbf{E}Y(1+o(1))$$

for m even. The contribution due to $s = 0$ is just the total number of m-cycles in all double 1-factors, divided by M. Because D only contains even cycles, this is 0 when m is odd. When m is even, it is easily calculated to be asymptotic to $\mathbf{E}Y/m$. This establishes (2.12).

The proof of (2.11) follows the proof of (2.12) by using a set of i_1 distinct cycles of length 3, i_2 of length 4, etc., where the cycles of the same length have been ordered. The two cases $s = 0$ and $s > 0$ can be treated independently for each cycle, and (2.11) follows. ∎

3 r-regular graphs

Here we show that the result in Section 2 implies contiguity between $\Omega_{2n,r}$ and $\Omega^+_{2n,r}$ in general ($r \geq 3$). In this we make use of the results in [13].

Theorem 3. *$\Omega_{2n,r}$ is contiguous with $\Omega^+_{2n,r}$.*

Proof. We show this by induction on r. The base case, $r = 3$, is implied by Theorem and Corollary 1.

Suppose that the claim is true for arbitrary $r \geq 3$. Let M be the number of 1-factors of $G \in \Omega_{2n,r+1}$. In [13], the last two authors show that:

$$\mathbf{P}\left(\epsilon < \frac{M}{\mathbf{E}M} < \frac{1}{\epsilon}\right) \to 1$$

as $\epsilon \to 0$ (slowly) and $n \to \infty$ (quickly).

It follows that $\Omega_{2n,r+1}$ is contiguous with the space $\Omega'_{2n,r+1}$, on the same underlying set, whose elements are generated by taking the union of a uniformly selected edge disjoint pair (D, M), where D is an r-regular graph and M is a perfect matching, each with vertex set $[2n]$. From the main result in [2], it is seen that each r-regular graph lies in

$$(\sqrt{2}+o(1))\left(\frac{2rn}{\mathrm{e}}\right)^{rn}(r!)^{-2n}\exp\left(-\frac{(r-1)^2}{4}-\frac{r-1}{2}-\frac{r^2}{2}\right)$$

such pairs. Thus $\Omega'_{2n,r+1}$ is contiguous with the space $\Omega''_{2n,r+1}$, on the same underlying set, whose elements are generated by first choosing $D \in \Omega_{2n,r}$, and then adding a uniformly random 1-factor of the complement of D.

By the inductive hypothesis, $\Omega''_{2n,r+1}$ is contiguous with the space whose elements are generated by taking an r-tuple of edge-disjoint perfect matchings $(M_1, \ldots, M_r)$ uniformly at random, and then adding a perfect matching M_{r+1}, uniformly at random from amongst all matchings edge-disjoint from $\cup_{i=1}^{r} M_i$. Again by the main theorem in [2], this space is contiguous with $\Omega^{+}_{2n,r}$. ∎

4 Triple 1-factors in bicoloured regular graphs

In this section we modify the method in Sections 2 and 3 so as to apply to bicoloured regular graphs. From Bender [1], the number M of labelled bicoloured cubic graphs on $2n$ vertices is asymptotically

$$M \sim \frac{(3n)!}{e^2 6^{2n}} \binom{2n}{n}. \tag{4.1}$$

Since bicoloured graphs contain no cycles of odd length, we alter the definition of X_i in this section.

Theorem 4. *Let $Y = Y_n$ denote the number of double 1-factors in a random bicoloured cubic graph on $2n$ vertices, and let X_i denote the number of cycles of length $2i$.*

$$\mathbf{E}Y \sim \frac{2^{2n+1}\pi n}{3^n\sqrt{3}e^{1/2}},$$

$$\frac{\mathbf{E}Y^2}{(\mathbf{E}Y)^2} \to \frac{16}{9e^{1/2}}.$$

Furthermore, if we define $\lambda_i = \frac{2^{2i-1}}{i}$ and $\mu_i = \frac{1}{i}$ for $i \geq 2$, then the hypotheses (1.1) – (1.10) of Theorem 1 all hold with the indices in X_i, λ_i and μ_i all shifted down by 1.

Proof. The colouring of the vertices can be selected in $\binom{2n}{n}$ ways. The number of perfect matchings of the vertices respecting the chosen colouring is $n!$. By the main theorem in [1], the number of ways to choose a second perfect matching, avoiding the edges in the first, is asymptotic to $n!/e$. This gives the asymptotic number of double 1-factors. The number of ways to complete a double 1-factor to a cubic graph is again given by the result of [1] to be asymptotic to $n!/e^2$. We obtain the claimed value of $\mathbf{E}Y$ by taking the product of these numbers and dividing by M.

It will next be shown that

$$\mathbf{E}Y^2 \sim (\mathbf{E}Y)^2 \frac{16}{9e^{1/2}}, \tag{4.2}$$

from which the second statement in the theorem follows. As in the proof of Theorem 2, we prove this by showing that total number T of ordered pairs of distinct double

1-factors in all bicoloured cubic graphs on $2n$ vertices satisfies

$$T \sim \frac{(2n)!n^{n+3/2}2^{2n+11/2}}{3^{n+1/2}e^{n+4}\sqrt{\pi}}. \tag{4.3}$$

The proof is similar to the proof of (2.3); the number of arrangements of paths and cycles in $H_1 \cap H_2$ is again given by (2.4), but in this case we have to consider 2-colouring the vertices of the paths and cycles. The cycles can be 2-coloured in 2^s ways, the $k-2i$ paths of odd length can be coloured in any of the 2^{k-2i} ways, and the $2i$ even-length paths can be coloured in $\binom{2i}{i}$ ways since the total number of vertices of each colour must be the same.

The choice of A_j and B_j is as before. Using the Theorem in [1], the number of completions to H_1 is equal to

$$a_1!(k-a_1)!e^{-\kappa_1}$$

where $\kappa_1 \to 1$ provided $a_1 \sim d \sim k/2 \to \infty$, and $\kappa_1 \geq 0$ always, with a_1 and d are defined as before.

The number of completions to H_2 is equal to

$$a_2!(k-a_2)!e^{-\kappa_2}$$

where $\kappa_2 \to 2$ provided $a_2 \sim d \sim d' \sim k/2 \to \infty$, and $\kappa_2 \geq 0$ always, where a_2 and d' are defined as before.

Finally to complete $H_1 \cup H_2$ to the graph G we must add a matching of the remaining $2n-2k$ vertices of degree 2 respecting the colouring. The number of completions is equal to

$$(n-k)!e^{-\kappa_3}$$

where $\kappa_3 \sim \frac{2n-3k+d}{n-k} \to 1$ provided $d \sim k/2$ and $n-k \to \infty$, and $\kappa_3 \geq 0$ always.

Thus we find that the expression in place of (2.10) is

$$\frac{n^{2n}2^{2n}3^3}{e^{3n}} \sum_{k,i,a_1,a_2,d,d'} \left(\frac{-2(n-k-i)}{n-i}\right)^s \frac{F}{s!e^{\kappa_1+\kappa_2+\kappa_3}} \tag{4.4}$$

where F is defined as in (2.10). The limiting behaviour of κ_1, κ_2 and κ_3 can now be determined as in Section 2. Hence (4.4) is asymptotic to

$$\frac{n^{3n}2^{5n+9/2}}{3^n e^{3n+2}(-2)^s s!},$$

and so the second statement in the theorem follows as before.

The hypothesis of Theorem 1 that the X_i are asymptotically independent Poisson with expectation λ_i is easily verified, and was used in [13].

The rest of the proof follows that of Theorem 2, noting that the contribution from cycles C of length $2i$ is

$$\frac{2^{2i}+1}{2i}\mathbf{E}Y(1+o(1))$$

for $s>0$ and $\frac{1}{2i}\mathbf{E}Y$ for $s=0$. ∎

We note again by Corollary 1, that the space of random r-regular bicoloured graphs is contiguous with the space generated by a random r-tuple of edge-disjoint bicoloured 1-factors. An r-regular bicoloured graph on $2n$ vertices is equivalent to a regular digraph, perhaps containing loops, on n vertices with in- and out-degrees all equal to r (an r-regular digraph). It follows that the space of random r-regular digraphs on n vertices with no loops or multiple edges is contiguous with the space generated by a random set of r edge-disjoint 1-regular digraphs, since for r-regular digraphs the probability of a loop or multiple edge is asymptotic to a non-zero constant (by the results of [[1]] for example). In the next section, we use this to show that a random simple r-regular digraph is almost surely strongly r-connected for $r\geq 2$.

5 The connectivity of regular digraphs

Recall that a digraph G is *strongly connected* if for any $u,v\in V(G)$, there is a directed $(u,v)-$path in G. For digraphs on at least $r+1$ vertices, G is *strongly r-connected* iff for any $W\subset V(G)$ with $|W|<r$, $G-W$ is strongly connected. A *simple* digraph is one with no loops or multiple edges.

Recently, Cooper [4] has shown that a random simple 2-regular digraph, G, is a.s. strongly 2-connected. To do this, he considered a directed multigraph F_2, formed by taking the union of two random permutations, where we use "permutation" to denote the corresponding 1-regular digraph. He showed that if a property P holds for F_2 with probability $1-o(n^{-1/2})$ then it holds for G with probability $1-o(1)$. This allowed him to prove his result concerning G by analysing F_2. The results of Section 4 allow us to use his methods to show that a.e. r-regular digraph is strongly r-connected for $r\geq 2$. Note on the other hand that a random 1-regular digraph is a.s. disconnected.

Theorem 5. *For any fixed $r\geq 2$, a random r-regular simple digraph is a.s. strongly r-connected .*

Proof. The case $r=2$ was proved in [4].

For $r\geq 3$, consider a directed multigraph, F_r, formed by taking the union of r independently and uniformly chosen permutations of $[n]=\{1,2,...,n\}$: $\pi_1,\dots,\pi_r$. It is straightforward to verify that F_r is simple with probability asymptotic to $\exp\left(-r-\binom{r}{2}\right)$.

Therefore, if a property P holds a.s. for F_r, then it holds a.s. for the union of a random set of r edge-disjoint permutations of $[n]$. Thus, by the results of Section 4, P holds a.s. for a random simple r-regular digraph on n vertices. To complete our proof then, it suffices to show that F_r is a.s. either non-simple or strongly r-connected.

Let Q be the number of partitions $S \cup T$ of $[n]$ with $|T| \geq r$ and $|N^+(S) \cap T| < r$ (where $N^+(S)$ is the outneighbourhood of S in F_r). Note that for $n \geq r+1$, F_r is strongly r-connected iff $Q = 0$. Let I be the indicator variable:

$$I = \begin{cases} 1, & F_r \text{ is simple,} \\ 0, & \text{otherwise.} \end{cases}$$

We will show that $\mathbf{E}(IQ) = o(1)$, which implies a.s. $IQ = 0$, completing the proof.

Note that since F_r is r-regular, in any partition $S \cup T$ of $[n]$, where $|T| \geq r$, $|N^+(S) \cap T| < r$, either $|S| \geq 2$ and $|T| \geq r+1$, or $I = 0$.

For $2 \leq j \leq n - r - 1$, consider any partition $[n] = S \cup T$ with $|S| = j$, and any $W = \{w_1, \ldots, w_{r-1}\} \subset T$. We will bound the number of permutations F_1 for which every (S, T) edge terminates in W.

Suppose that there are k such edges. Note that $k \leq r-1$ and that there must be exactly k edges from T to S. Upon choosing these $2k$ edges, the remainder of the graph consists of two permutations, one on S and the other on T. There is a one-to-one correspondence between the permutation on S (resp. T) and a permutation of $[j-k]$ (resp. $[n-j-k]$). Thus, the number of choices for F_1 is at most:

$$\left(\binom{j}{k}\binom{r-1}{k}k!\right)\left(\binom{j}{k}\binom{n-j}{k}k!\right)(j-k)!(n-j-k)! = O(j^k j!(n-j)!).$$

Therefore,

$$\begin{aligned} \mathbf{E}(IQ) &= O(1) \times \sum_{j=2}^{n-r-1} \binom{n}{j}\binom{n-j}{r-1}\left(\frac{j^k j!(n-j)!}{n!}\right)^r \\ &= O\left(n^{-(r-1)}\right). \quad \blacksquare \end{aligned}$$

References

[1] E. A. Bender, The asymptotic number of non-negative integer matrices with given row and column sums, *Discrete Math.* **10** (1974), 217–223.

[2] E. A. Bender and E. R. Canfield, The asymptotic number of labeled graphs with given degree sequences, *J. Combin. Theory Ser. A* **24** (1978), 296–307.

[3] B. Bollobás and B. D. McKay, The number of matchings in random regular graphs and bipartite graphs, *J. Combin. Theory Ser. B* **41** (1986), 80–91.

[4] C. Cooper, A note on the connectivity of 2-regular digraphs, *Random Structures and Algorithms* **4** (1993), 469–472.

[5] C. Cooper, A. Frieze and M. Molloy. Hamilton cycles in random regular digraphs, *Combinatorics, Probability and Computing* **3** (1994), 39 – 50.

[6] A. Frieze, M. Jerrum, M. Molloy, R. W. Robinson and N. C. Wormald, Generating and counting hamilton cycles in random regular graphs, *Journal of Algorithms* (to appear).

[7] S. Janson, Random regular graphs: Asymptotic distributions and contiguity, *Combinatorics, Probability and Computing* (to appear).

[8] L. Le Cam, Locally asymptotically normal families of distributions, *Univ. of California Publ. in Statistics* **3** (1960), 37 – 98.

[9] M. Molloy and B. Reed, The dominating number of a random cubic graph, *Random Structures and Algorithms* **7** (1995), 209 – 221.

[10] H. Robalewska, 2-factors in random regular graphs, *J. Graph Theory* (to appear).

[11] R. W. Robinson and N. C. Wormald, Existence of long cycles in random cubic graphs, in *Enumeration and Design* (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, Toronto (1984), 251–270.

[12] R. W. Robinson and N. C. Wormald, Almost all cubic graphs are hamiltonian, *Random Structures and Algorithms* **3** (1992), 117–125.

[13] R. W. Robinson and N. C. Wormald, Almost all regular graphs are hamiltonian, *Random Structures and Algorithms* **5** (1994), 363 – 374.

[14] G. Roussas, *Contiguity of probability measures: some applications in statistics*, Cambridge Univ. Press, Cambridge, 1972.