CSCI 2670
Introduction to Theory of Computing

November 2, 2004
Agenda

• Last week
  - Decidability & undecidability

• Today
  - One more undecidability proof
  - Reductions (Section 5.3)

• This week
  - Section 5.3
  - Section 6.3 & part of 6.4
Announcements

• Quiz tomorrow
  - Decidable and undecidable languages (hints will be provided)
  - Countable sets
• Homework due next Tuesday (11/9)
  - 5.4, 5.5, 5.7, 5.20, 6.3, 6.16
• Second midterm is next week
  - Chapters 3 & 4, parts of 5 & 6
Another undecidable language

Let \( \text{REGULAR}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language} \} \)

**Theorem:** \( \text{REGULAR}_{\text{TM}} \) is undecidable

**Proof:** Assume \( R \) decides \( \text{REGULAR}_{\text{TM}} \) and use \( R \) to decide \( A_{\text{TM}} \) (reduce the \( A_{\text{TM}} \) problem to the \( \text{REGULAR}_{\text{TM}} \) problem).

As before, make a new TM, \( M_2 \), that accepts a regular language iff \( M \) accepts \( w \).
Proof (cont.)

Consider the following TM $S$ = “On input $<M,w>$

1. Construct the following TM $M_2$
   $M_2$ = “On input $x$
   1. If $x = 0^n1^n$ for some $n$, accept
   2. Otherwise, run $M$ on $w$. If $M$ accepts $w$, accept”

2. Run $R$ on $M_2$ (accepts iff $L(M_2)$ is a RL)

3. If $R$ accepts, accept; if $R$ rejects, reject”

$S$ decides $A_{TM}$ if $R$ decides $REGULAR_{TM}$
Insight

- The TM $M_2$ is specially designed to be regular if and only if $M$ accepts $w$
- Then call TM that decides $\text{REGULAR}_{TM}$ on $M_2$
Rice’s theorem

• Determining whether a TM satisfies any non-trivial property is undecidable
  - A property is non-trivial if:
    1. It depends only on the language of $M$, and
    2. Some, but not all, Turing machines have the property
  - Examples: Is $L(M)$ regular? A CFG? Finite?
Proof of Rice’s theorem

• Assume there is some decidable non-trivial property $P$ for Turing machines
  - Assume TM’s that accept $\emptyset$ do not satisfy $P$
    • If they do, just consider $\neg P$
Proof of Rice’s theorem

• Let TM B decide P
  – On input <M>, B accepts iff TM M has property P

• Let MP be a TM that satisfies P
  – Since P is non-trivial, there is some MP satisfying P

• Use B and MP to decide $A_{TM}$
  – Create a new TM S that decides $A_{TM}$ using B and MP
A_{TM} decider using MP

\[ S = \text{"On input } \langle M, w \rangle \text{"} \]

1. Create the following TM \( N \)
   \[ N = \text{"On input } x \text{"} \]
   1. Run \( M \) on \( w \) until it accepts
   2. If \( M \) accepts \( w \), run \( MP \) on \( x \)
   3. If \( MP \) accepts \( x \), accept; if \( MP \) rejects \( x \), reject

2. Run \( B \) on \( \langle N(\langle M, w \rangle) \rangle \)
   If \( B \) accepts, accept; if \( B \) rejects, reject

\[ L(N) = L(MP) \text{ if } M \text{ accepts } w; \text{ otherwise } L(N) = \emptyset \]

\( S \) decides \( A_{TM} \) if \( B \) decides TM’s satisfying \( P \)