CSCI 2670
Introduction to Theory of Computing

October 20, 2004
Agenda

• Yesterday
  - Some decidable problems involving regular languages and finite automata
  - One undecidable language
    • EQ_{CFG}

• Today
  - More undecidable languages
  - Techniques for showing a language is not decidable
Announcements

• Quiz tomorrow
  - Enumerators, definition of algorithm, decidable languages

• No tutorial tomorrow
  - Extra office hours today (3:00 – 5:00)
Relationship of classes of languages

- Regular
- Context-free
- Decidable
- Turing-recognizable
Turing machine acceptance problem

- Consider the following language
  \[ A_{TM} = \{<M,w> \mid M \text{ is a TM that accepts } w \} \]

**Theorem:** \( A_{TM} \) is Turing-recognizable

**Theorem:** \( A_{TM} \) is undecidable

**Proof:** The universal Turing machine recognizes, but does not decide, \( A_{TM} \)
The universal Turing machine

$U = \text{"On input } <M, w>, \text{ where } M \text{ is a TM and } w \text{ is a string:\"}

1. Simulate $M$ on input $w$
2. If $M$ ever enters its accept state, accept
3. If $M$ ever enters its reject state, reject"
Why can’t U decide $A_{TM}$?

- Intuitively, if $M$ never halts on $w$, then $U$ never halts on $<M,w>$
- This is also known as the halting problem
  - Given a TM $M$ and a string $w$, does $M$ halt on input $w$?
  - Undecidable
- We will prove this more rigorously later
  - Need some new tools for proving properties of languages
Comparing the size of infinite sets

• Given two infinite sets $A$ and $B$, is there any way of determining if $|A|>|B|$?
  - Yes!

• Diagonalization
Functional correspondence

• Let $f$ be a function from $A$ to $B$

• $f$ is called **one-to-one** if ...
  - $f(a) \neq f(b)$ whenever $a \neq b$

• $f$ is called **onto** if ...
  - For every $b \in B$, there is some $a \in A$ such that $f(a) = f(b)$

• $f$ is called a **correspondence** if it is one-to-one and onto
  - A correspondence is a way to pair elements of the two sets
Example correspondence

- Consider $f: \mathbb{Z}^* \rightarrow P$, where $\mathbb{Z}^* = \{0,1,2,...\}$ and $P = \{\text{positive squares}\}$
  - $f(x) = (x+1)^2$

- Is $f$ one-to-one?
  - Yes

- Is $f$ onto?
  - Yes

- Therefore $|\mathbb{Z}| = |P|$
Countable sets

- Let $N = \{1, 2, 3, \ldots\}$ be the set of natural numbers
- The set $A$ is countable if ...
  - $A$ is finite, or
  - $|A| = |N|$
- Some example of countable sets
  - Integers
  - $\{x \mid x \in \mathbb{Z} \text{ and } x = 1 \pmod{3}\}$
The positive rational numbers

- Is $Q = \{m/n \mid m, n \in \mathbb{N}\}$ countable?
  - Yes!

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The real numbers

• Is $\mathbb{R}^+$ (the set of positive real numbers) countable?
  - No!

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$X = 4.1337...$

Diagonalization