CSCI 2670
Introduction to Theory of Computing

October 21, 2004
Agenda

• **Yesterday**
  - Some decidable problems involving regular languages and finite automata
  - One undecidable language
    • $\text{EQ}_{\text{CFG}}$

• **Today**
  - More undecidable languages
  - Techniques for showing a language is not decidable


Agenda

• Today
  - Finish Chapter 4

• Next week
  - Go over Turing machines and decidability
  - Begin Chapter 5

• No tutorial today
Countable sets

• Let $N = \{1, 2, 3, \ldots\}$ be the set of natural numbers

• The set $A$ is countable if ...
  - $A$ is finite, or
  - $|A| = |N|$

• Some example of countable sets
  - Integers
  - $\{x \mid x \in \mathbb{Z} \text{ and } x = 1 \pmod{3}\}$
The positive rational numbers

- Is $Q = \{ \frac{m}{n} \mid m, n \in \mathbb{N} \}$ countable?
  - Yes!

| 1/1 | 1/2 | 1/3 | 1/4 | 1/5 | 2/1 | 2/2 | 2/3 | 2/4 | 2/5 | 3/1 | 3/2 | 3/3 | 3/4 | 3/5 | 4/1 | 4/2 | 4/3 | 4/4 | 4/5 | 5/1 | 5/2 | 5/3 | 5/4 | 5/5 | Etc... |
The real numbers

• Is $R^+$ (the set of positive real numbers) countable?
  - No!

<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
<th>X = 4.1337...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.56439...</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.23891...</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7.42210...</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.22266...</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.16982...</td>
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</tbody>
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Diagonalization
The set of all infinite binary strings

- Is the set of all infinite binary strings countable?
  - No
  - Diagonalization also works to prove this is not countable
Is the set of all languages in $\Sigma^*$ countable?

• No
  - This set has the same cardinality as the set of binary strings

$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

$A = \{ a, \text{ab}, \text{aaa}, \ldots\}$

$\chi_A = 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ldots$
Is the set of all TM’s countable?

• Yes
• Since all Turing machines can be represented by a string, the set of all Turing machines are countable

Theorem: Some languages are not Turing-recognizable

Proof: There are more languages than there are Turing machines
Undecidability of halting problem

Theorem: \( A_{TM} \) is undecidable

Proof: (By contradiction) Assume \( A_{TM} \) is decidable and let \( H \) be a decider for \( A_{TM} \)

\[
H(<M,w>) = \begin{cases} 
\text{accept} & \text{if } M \text{ accepts } w \\
\text{reject} & \text{if } M \text{ does not accept } w 
\end{cases}
\]
Proof (cont.)

• Consider the TM $D$ that submits the string $<M>$ as input to the TM $M$

$$D = \text{"On input } <M>, \text{ where } M \text{ is a TM:}$$

1. Run $H$ on input $<M,<M>>$
2. If $H$ accepts $<M,<M>>$, reject
3. If $H$ rejects $<M,<M>>$, accept
   - Since $H$ is a decider, it must accept or reject
   - Therefore, $D$ is a decider as well
Proof (cont.)

• What happens if $D'$ input is $<D>$?

$$D(<D>) = \begin{cases} 
\text{reject} & \text{if } D \text{ accepts } <D> \\
\text{accept} & \text{if } D \text{ does not accept } <D>
\end{cases}$$

• $D$ cannot exist!

• Therefore, $H$ cannot exist – i.e., $A_{TM}$ is undecidable
Have a fabulous weekend