(1) Claim: Given any alphabet $\Sigma$, the language $\Sigma^*$ is countable.
   a. (2 points) Formally explain what it means for $\Sigma^*$ to be countable.

   The set $\Sigma^*$ is countable if (1) the size of the set is finite or (2) the size is the same as the size of the natural numbers, $\mathbb{N}$. Since $\Sigma^*$ is an infinite set, you can determine that it is the same size as $\mathbb{N}$ by finding a correspondence – i.e., a function that is one-to-one and onto – between $\mathbb{N}$ and $\Sigma^*$.

   b. (3 points) Prove this claim is true.

   List the strings of $\Sigma^*$ in “alphabetical” order starting with the shortest strings and working toward longer strings. Thus, the first string in the list will have length 0, then next $n$ strings in the list will have length 1, then next $n^2$ strings will have length 2, and so on. Define the function $f$ so that $f(i) = \text{the } i^{\text{th}} \text{ string in the list}$. This function is clearly one-to-one and onto.
Consider the language $\text{NOHALT}_\text{TM}$. This language is undecidable.

a. (2 points) Explain how you would prove this is an undecidable language using the fact that $A_\text{TM}$ is known to be undecidable.

Assume $\text{NOHALT}_\text{TM}$ is decidable and is decided by Turing machine $M$. Then create another Turing machine $S$ that calls $M$ and decides the language $A_\text{TM}$. Since $A_\text{TM}$ is known to be undecidable, conclude that there is no $M$ that decides $\text{NOHALT}_\text{TM}$.

b. (3 points) Using the strategy you described in part a, prove $\text{NOHALT}_\text{TM}$ is undecidable.

Assume $\text{NOHALT}_\text{TM}$ is decided by Turing machine $N$—i.e., $N$ accepts all strings $<M,w>$ where $M$ is a Turing machine that does not halt when $M$ runs on input string $w$ and $N$ rejects all other inputs. Consider the following Turing machine $S$.

$S = \text{“On input } <M,w>\text{ “}$
$\quad \text{Run } N \text{ on input } <M,w>$
$\quad \text{If } N \text{ rejects, then accept}$
$\quad \text{If } N \text{ accepts, then run } M \text{ on } w \text{ and return the result}$
$\quad \text{(accept if } M \text{ accepts } w \text{ and reject if } M \text{ rejects } w)$

The Turing machine $S$ decides the language $A_\text{TM}$, which is undecidable. This is a contradiction. Therefore, $\text{NOHALT}_\text{TM}$ must be undecidable.